

On generalized Lattès maps

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Abstract

We introduce a class of rational functions $A : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ which can be considered as a natural extension of the class of Lattès maps and establish basic properties of functions from this class.

1 Introduction

Lattès maps are rational functions $A : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ of degree at least two which can be characterized in one of the following equivalent ways (see [10]). First, a Lattès map A can be defined by the condition that there exist a compact Riemann surface \mathcal{R} of genus one and holomorphic maps $B : \mathcal{R} \rightarrow \mathcal{R}$ and $\pi : \mathcal{R} \rightarrow \mathbb{CP}^1$ such that the diagram

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{B} & \mathcal{R} \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array} \quad (1)$$

is commutative.

This condition can be replaced by the apparently stronger condition that π in the diagram above is the quotient map $\pi : \mathcal{R} \rightarrow \mathcal{R}/\Gamma$ for some finite subgroup Γ of the automorphism group $\text{Aut}(\mathbb{C})$. Equivalently, by passing to the universal covering we may assume that the Riemann surface \mathcal{R} in (1) is \mathbb{C} , the map B is an automorphism of \mathbb{C} , and π is the quotient map $\pi : \mathbb{C} \rightarrow \mathbb{C}/\Gamma$ for some subgroup Γ of $\text{Aut}(\mathbb{C})$ acting properly discontinuously on \mathbb{C} .

Finally, Lattès maps may be characterized in terms of their ramification. This characterization uses the notion of *orbifold* on \mathbb{CP}^1 . By definition, an orbifold \mathcal{O} on \mathbb{CP}^1 is a ramification function $\nu : \mathbb{CP}^1 \rightarrow \mathbb{N}$ which takes the value $\nu(z) = 1$ except a finite set of points. We always will assume that considered orbifolds are *good* meaning that we forbid \mathcal{O} to have exactly one point with $\nu(z) \neq 1$ or two such points z_1, z_2 with $\nu(z_1) \neq \nu(z_2)$. A rational function f is called a *covering map* $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ between orbifolds with ramifications functions ν_1 and ν_2 if for any $z \in \mathbb{CP}^1$ the equality

$$\nu_2(f(z)) = \nu_1(z) \deg_z f \quad (2)$$

holds. In these terms, a Lattès map can be defined as a rational function A such that $A : \mathcal{O} \rightarrow \mathcal{O}$ is a covering self-map for some orbifold \mathcal{O} .

In the recent paper [13] a class of rational functions A satisfying condition (1) under the assumption that \mathcal{R} is the *Riemann sphere* was considered. In particular, it was shown in [13] that under certain restrictions such functions possess a number of remarkable properties similar to the ones of Lattès maps. In this paper we show that this analogy is quite deep and construct, modifying the approach of [13], a unified theory which equally fits the classical Lattès maps and functions studied in [13].

Let \mathcal{R} be a compact Riemann surface of genus zero or one and A, B, π holomorphic maps satisfying (1). We say that B and π have a *non-trivial common compositional right factor* if there exist a compact Riemann surface \mathcal{R}' and a holomorphic map $w : \mathcal{R} \rightarrow \mathcal{R}'$ of degree at least two such that

$$B = \tilde{B} \circ w, \quad \pi = \tilde{\pi} \circ w \quad (3)$$

for some holomorphic maps $\tilde{B} : \mathcal{R}' \rightarrow \mathcal{R}$ and $\tilde{\pi} : \mathcal{R}' \rightarrow \mathbb{CP}^1$. It is easy to see that if (3) holds, then the maps

$$B' = w \circ \tilde{B}, \quad \pi' = \tilde{\pi},$$

and A satisfy (1) for $\mathcal{R} = \mathcal{R}'$. Furthermore, $\deg \pi' < \deg \pi$ since $\deg w > 1$. If the maps π' and B' still have a non-trivial common compositional factor we can repeat the above transformation. In this way, after a finite number of steps we will arrive to maps B_0 and π_0 such that A, B_0, π_0 satisfy (1) while B_0 and π_0 have no non-trivial common compositional right factor. Depending on the degree of the map π_0 , this construction gives rise to two important notions which can be described as follows.

In case if $\deg \pi_0 = 1$, the genus of the initial surface \mathcal{R} is necessarily zero and the initial functions A and B can be considered as *equivalent* with respect to the equivalence relation \sim defined as follows. Let B be a rational function. For any decomposition $B = V \circ U$ of B into a composition of rational functions U and V the rational function $\tilde{B} = U \circ V$ is called an elementary transformation of B , and rational functions B and A are called equivalent if there exists a chain of elementary transformations between B and A . Since for any Möbius transformation W the equality

$$B = (B \circ W) \circ W^{-1}$$

holds, each equivalence class of \sim is a union of conjugacy classes. Moreover, one can show that the equivalence class of B contains infinitely many conjugacy classes if and only if B is a flexible Lattès function (see [17]). Thus, the relation \sim can be considered as a weaker form of the classical conjugacy relation.

In case if $\deg \pi_0 > 1$, we arrive to the notion of a generalized Lattès map. In more details, a generalized Lattès map A can be defined as a rational function A such that (1) holds for some compact Riemann surface \mathcal{R} of genus zero or one and holomorphic maps B and π having no non-trivial common compositional

right factor. Equivalently, this class of rational functions can be described by any of the conditions listed in the theorem below.

Theorem 1.1. *Let A be a rational function of degree at least two. Then the following conditions are equivalent.*

1. *There exist a compact Riemann surface \mathcal{R} of genus zero or one and holomorphic maps $B : \mathcal{R} \rightarrow \mathcal{R}$ and $\pi : \mathcal{R} \rightarrow \mathbb{CP}^1$ of degree at least two such that the diagram*

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{B} & \mathcal{R} \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array} \quad (4)$$

is commutative and B and π have no non-trivial common compositional right factor.

2. *There exist a compact Riemann surface \mathcal{R} of genus zero or one, a finite non-trivial group $\Gamma \subseteq \text{Aut}(\mathcal{R})$, and a holomorphic map $B : \mathcal{R} \rightarrow \mathcal{R}$ such that the diagram*

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{B} & \mathcal{R} \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1, \end{array} \quad (5)$$

where $\pi : \mathcal{R} \rightarrow \mathcal{R}/\Gamma$ stands for the quotient map, is commutative and for any $\sigma \in \Gamma$ the equality

$$B \circ \sigma = \varphi(\sigma) \circ B \quad (6)$$

holds for some automorphism $\varphi : \Gamma \rightarrow \Gamma$.

3. *There exist a simply connected Riemann surface \mathcal{R} equal either to \mathbb{C} or to \mathbb{CP}^1 , a non-trivial group $\Gamma \subseteq \text{Aut}(\mathcal{R})$ acting properly discontinuously on \mathcal{R} , and a holomorphic map $B : \mathcal{R} \rightarrow \mathcal{R}$ such that the diagram*

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{B} & \mathcal{R} \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1, \end{array} \quad (7)$$

where $\pi : \mathcal{R} \rightarrow \mathcal{R}/\Gamma$ stands for the quotient map, is commutative and for any $\sigma \in \Gamma$ the equality

$$B \circ \sigma = \varphi(\sigma) \circ B \quad (8)$$

holds for some monomorphism $\varphi : \Gamma \rightarrow \Gamma$.

4. *There exists a good orbifold Θ distinct from the non-ramified sphere such that for any $z \in \mathbb{CP}^1$ the equality*

$$\nu(A(z)) = \nu(z) \text{GCD}(\deg_z A, \nu(A(z))) \quad (9)$$

holds.

Groups and orbifolds which actually may satisfy the conditions listed in Theorem 1.1 can be described explicitly. Namely, if the Riemann surface \mathcal{R} in the second condition has genus one, then the corresponding group $\Gamma \subseteq \text{Aut}(\mathcal{R})$ is necessarily a cyclic group C_n with n equal 2,3,4, or 6, while if $\mathcal{R} = \mathbb{CP}^1$ this group is one of the five finite subgroups $C_n, D_{2n}, A_4, S_4, A_5$ of $\text{Aut}(\mathbb{CP}^1)$. Accordingly, if the Riemann surface \mathcal{R} in the third condition is \mathbb{CP}^1 , then the corresponding group $\Gamma \subseteq \text{Aut}(\mathbb{C})$ is generated by translations of \mathbb{C} by elements of some lattice $L \subset \mathbb{C}$ of rank two and the transformation $z \rightarrow \varepsilon z$, where ε is an n th root of unity, with n equal to 2,3,4, or 6, such that $\varepsilon L = L$. On the other hand, if $\mathcal{R} = \mathbb{CP}^1$, then the third condition reduces to the second one, and Γ is a finite subgroup of $\text{Aut}(\mathbb{CP}^1)$.

It is easy to see that whenever $A : \mathcal{O} \rightarrow \mathcal{O}$ is covering map between orbifolds condition (9) also holds although the inverse is not true in general. Moreover, one can show that for any covering map $A : \mathcal{O} \rightarrow \mathcal{O}$ the *Euler characteristic*

$$\chi(\mathcal{O}) = 2 + \sum_{z \in R} \left(\frac{1}{\nu(z)} - 1 \right) \quad (10)$$

of \mathcal{O} equals zero, implying that the ramification collection of \mathcal{O} is one of the triples

$$\{2, 2, 2\}, \quad \{3, 3, 3\}, \quad \{2, 4, 4\}, \quad \{2, 3, 6\}.$$

On the other hand, condition (9) implies that the Euler characteristic of \mathcal{O} is *non-negative*. Thus, in addition to the above collections we should consider the collections

$$\{n, n\}, \quad n \geq 2, \quad \{2, 2, n\}, \quad n \geq 2, \quad \{2, 3, 3\}, \quad \{2, 3, 4\}, \quad \{2, 3, 5\}$$

corresponding to orbifolds of positive Euler characteristic. Notice that the Euler characteristic of the sphere considered as a non-ramified orbifold is also positive but for this orbifold condition (9) holds for *any* rational function A . Thus, the exclusion of this case corresponds to the assumptions that $\deg \pi \geq 2$ in the first condition and $\Gamma \neq \{e\}$ in the second and the third conditions.

We will call a rational function A satisfying any of the conditions provided by Theorem 1.1 a *generalized Lattès functions*, and the goal of the paper is to establish basic properties of these functions, in particular to prove Theorem 1.1. The paper is organized as follows.

In the second section we recall main technical results of [13] about Riemann surfaces orbifolds and different kinds of maps between orbifolds. In the third section we prove Theorem 1.1 and provide a description of the general structure of solutions of (1). We also study compositional properties of generalized Lattès function. In more details, recall that a rational function A is called indecomposable if the equality $A = U \circ V$, where U and V are rational functions, implies that at least one of the functions U and V has degree one. Clearly, any rational function A can be decomposed into a composition

$$A = A_1 \circ A_2 \circ \cdots \circ A_l \quad (11)$$

of indecomposable rational functions of degree at least two; such decompositions are called maximal. For a given orbifold \mathcal{O} denote by $\mathcal{E}(\mathcal{O})$ the set of all rational functions satisfying (9). In this notation, the main result of the third section about decompositions of generalized Lattès functions can be formulated as follows: for any $U, V \in \mathcal{E}(\mathcal{O})$ the composition $U \circ V$ also belongs to $\mathcal{E}(\mathcal{O})$, and, whenever the collection of ramification indices of \mathcal{O} is distinct from $\{2, 2, 2, 2\}$, any $A \in \mathcal{E}(\mathcal{O})$ has maximal decomposition (11) whose elements belong to $\mathcal{E}(\mathcal{O})$.

In the fourth section we investigate the following problem: given a generalized Lattès map A , what are possible \mathcal{O} for which (9) is satisfied? For classical Lattès maps, the orbifold \mathcal{O} such that (2) holds is defined in a unique way by the dynamical properties of A . For generalized Lattès maps there might be several orbifolds \mathcal{O} satisfying (9). For example, for $A = z^{\pm n}$ equality (9) is satisfied for any orbifold \mathcal{O} defined by the conditions

$$\nu(0) = \nu(\infty) = m, \quad \text{GCD}(n, m) = 1.$$

We show however that, whenever A is not conjugated to $z^{\pm n}$ or $\pm T_n$, there exists an orbifold satisfying (9) and possessing a natural “maximality” property among all orbifolds satisfying (9). The proof is based on the link between generalized Lattès maps and functions satisfying (6), and involves the study of the group $\mathcal{G}(F)$, associated with a rational function F , consisting of all $\mu \in \text{Aut}(\mathbb{CP}^1)$ such that

$$F \circ \sigma = \nu_\sigma \circ F$$

for some $\nu_\sigma \in \text{Aut}(\mathbb{CP}^1)$.

Finally, the fifth section is devoted to finding explicit formulas for generalized Lattès maps. Here we focus on the case of generalized Lattès maps which are not classical Lattès maps, that is on functions satisfying (9) for some orbifold \mathcal{O} with $\chi(\mathcal{O}) > 0$. In particular, we describe explicitly the sets $\mathcal{E}(\mathcal{O})$ for orbifolds \mathcal{O} whose collection of ramification indices is $\{n, n\}$, $n \geq 2$, or $\{2, 2, n\}$, $n > 2$. We also describe all *polynomials* satisfying (9) for some \mathcal{O} with $\chi(\mathcal{O}) > 0$. In the general case we relate the problem with the problem of describing of rational functions commuting with finite automorphism groups of $\text{Aut}(\mathbb{CP}^1)$. We recall a description of such functions obtained by Doyle and McMullen ([3]), and give examples of practical calculations of corresponding generalized Lattès maps of small degrees. In particular, we construct a one-parameter family of generalized Lattès maps of degree six

$$f_a(z) = z \left(\frac{(a-1)^4 z^2 - 2(a-1)(a^3 - 3a^2 - 9a - 21)z + (a-7)(a+1)^3}{(a+7)(a-1)^3 z^2 - 2(a+1)(a^3 + 3a^2 - 9a + 21)z + (a+1)^4} \right)^3$$

satisfying (9) for \mathcal{O} defined by the equalities

$$\nu(0) = 3, \quad \nu(1) = 2, \quad \nu(\infty) = 3.$$

2 Orbifolds and maps between orbifolds

In this section we recall basic definitions concerning Riemann surface orbifolds (see e. g. Appendix E of [11]) and overview main technical results from the paper [13] concerning holomorphic and minimal holomorphic maps between orbifolds.

A Riemann surface orbifold is a pair $\mathcal{O} = (R, \nu)$ consisting of a Riemann surface R and a ramification function $\nu : R \rightarrow \mathbb{N}$ which takes the value $\nu(z) = 1$ except at isolated points. The Euler characteristic of an orbifold $\mathcal{O} = (R, \nu)$ is defined by the formula

$$\chi(\mathcal{O}) = \chi(R) + \sum_{z \in R} \left(\frac{1}{\nu(z)} - 1 \right), \quad (12)$$

where $\chi(R)$ is the Euler characteristic of R . For an orbifold $\mathcal{O} = (R, \nu)$ denote by $c(\mathcal{O})$ the set of ramified points of \mathcal{O} ,

$$c(\mathcal{O}) = \{z_1, z_2, \dots, z_s, \dots\} = \{z \in R \mid \nu(z) > 1\},$$

and set

$$\nu(\mathcal{O}) = \{\nu(z_1), \nu(z_2), \dots, \nu(z_s), \dots\}.$$

If R_1, R_2 are Riemann surfaces provided with ramification functions ν_1, ν_2 and $f : R_1 \rightarrow R_2$ is a holomorphic branched covering map, then f is called a *covering map* $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ *between orbifolds* $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$ if for any $z \in R_1$ the equality

$$\nu_2(f(z)) = \nu_1(z) \deg_z f \quad (13)$$

holds, where $\deg_z f$ stands for the local degree of f at the point z . If for any $z \in R_1$ instead of equality (13) a weaker condition

$$\nu_2(f(z)) \mid \nu_1(z) \deg_z f \quad (14)$$

holds, then f is called a *holomorphic map* $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ *between orbifolds* \mathcal{O}_1 and \mathcal{O}_2 .

A *universal covering* of an orbifold \mathcal{O} is a covering map between orbifolds $\theta_{\mathcal{O}} : \tilde{\mathcal{O}} \rightarrow \mathcal{O}$ such that \tilde{R} is simply connected and $\tilde{\nu}(z) \equiv 1$. If $\theta_{\mathcal{O}}$ is such a map, then there exists a group $\Gamma_{\mathcal{O}}$ of conformal automorphisms of \tilde{R} such that the equality $\theta_{\mathcal{O}}(z_1) = \theta_{\mathcal{O}}(z_2)$ holds for $z_1, z_2 \in \tilde{R}$ if and only if $z_1 = \sigma(z_2)$ for some $\sigma \in \Gamma_{\mathcal{O}}$. A universal covering exists and is unique up to a conformal isomorphism of \tilde{R} , unless \mathcal{O} is the Riemann sphere with one ramified point or with two ramified points z_1, z_2 such that $\nu(z_1) \neq \nu(z_2)$. Furthermore, $\tilde{R} = \mathbb{D}$ if and only if $\chi(\mathcal{O}) < 0$, $\tilde{R} = \mathbb{C}$ if and only if $\chi(\mathcal{O}) = 0$, and $\tilde{R} = \mathbb{CP}^1$ if and only if $\chi(\mathcal{O}) > 0$ (see [11], Appendix E, and [6], Section IV.9.12).

Covering maps between orbifolds lift to isomorphisms between their universal coverings. More generally, for holomorphic maps the following proposition holds (see [13], Proposition 3.1).

Proposition 2.1. *Let $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ be a holomorphic map between orbifolds. Then for any choice of $\theta_{\mathcal{O}_1}$ and $\theta_{\mathcal{O}_2}$ there exist a holomorphic map $F : \widetilde{\mathcal{O}}_1 \rightarrow \widetilde{\mathcal{O}}_2$ and a homomorphism $\varphi : \Gamma_{\mathcal{O}_1} \rightarrow \Gamma_{\mathcal{O}_2}$ such that the diagram*

$$\begin{array}{ccc} \widetilde{\mathcal{O}}_1 & \xrightarrow{F} & \widetilde{\mathcal{O}}_2 \\ \downarrow \theta_{\mathcal{O}_1} & & \downarrow \theta_{\mathcal{O}_2} \\ \mathcal{O}_1 & \xrightarrow{f} & \mathcal{O}_2 \end{array} \quad (15)$$

is commutative and for any $\sigma \in \Gamma_{\mathcal{O}_1}$ the equality

$$F \circ \sigma = \varphi(\sigma) \circ F \quad (16)$$

holds. The map F is defined by $\theta_{\mathcal{O}_1}$, $\theta_{\mathcal{O}_2}$, and f uniquely up to a transformation $F \rightarrow g \circ F$, where $g \in \Gamma_{\mathcal{O}_2}$. In the other direction, for any holomorphic map $F : \widetilde{\mathcal{O}}_1 \rightarrow \widetilde{\mathcal{O}}_2$ which satisfies (16) for some homomorphism $\varphi : \Gamma_{\mathcal{O}_1} \rightarrow \Gamma_{\mathcal{O}_2}$ there exists a uniquely defined holomorphic map between orbifolds $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ such that diagram (15) is commutative. The holomorphic map F is an isomorphism if and only if f is a covering map between orbifolds. \square

If $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ is a covering map between orbifolds with compact supports, then the Riemann-Hurwitz formula implies that

$$\chi(\mathcal{O}_1) = d\chi(\mathcal{O}_2), \quad (17)$$

where $d = \deg f$. For holomorphic maps the following statement is true (see [13], Proposition 3.2).

Proposition 2.2. *Let $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ be a holomorphic map between orbifolds with compact supports. Then*

$$\chi(\mathcal{O}_1) \leq \chi(\mathcal{O}_2) \deg f, \quad (18)$$

and the equality holds if and only if $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ is a covering map between orbifolds. \square

Let R_1, R_2 be Riemann surfaces and $f : R_1 \rightarrow R_2$ a holomorphic branched covering map. Assume that R_2 is provided with ramification function ν_2 . In order to define a ramification function ν_1 on R_1 so that f would be a holomorphic map between orbifolds $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$ we must satisfy condition (14), and it is easy to see that for any $z \in R_1$ a minimal possible value for $\nu_1(z)$ is defined by the equality

$$\nu_2(f(z)) = \nu_1(z) \text{GCD}(\deg_z f, \nu_2(f(z))). \quad (19)$$

In case if (19) is satisfied for any $z \in R_1$ we say that f is a *minimal holomorphic map between orbifolds* $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$.

It follows from the definition that for any orbifold $\mathcal{O} = (R, \nu)$ and holomorphic branched covering map $f : R' \rightarrow R$ there exists a unique orbifold structure ν' on R' such that f becomes a minimal holomorphic map between orbifolds. We will denote the corresponding orbifold by $f^*\mathcal{O}$. Notice that any covering map between orbifolds $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ is a minimal holomorphic map. In particular, $\mathcal{O}_1 = f^*\mathcal{O}_2$. For orbifolds \mathcal{O}_1 and \mathcal{O}_2 we will write

$$\nu(\mathcal{O}_1) \leq \nu(\mathcal{O}_2) \quad (20)$$

if for any $x \in c(\mathcal{O}_1)$ there exists $y \in c(\mathcal{O}_2)$ such that $\nu(x) \mid \nu(y)$. Clearly, the condition that $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ is a minimal holomorphic map between orbifolds implies condition (20).

Minimal holomorphic maps between orbifolds possess the following fundamental property with respect to compositions of maps (see [13], Theorem 4.1).

Theorem 2.3. *Let $f : R'' \rightarrow R'$ and $g : R' \rightarrow R$ be holomorphic branched covering maps, and $\mathcal{O} = (R, \nu)$ an orbifold. Then*

$$(g \circ f)^*\mathcal{O} = f^*(g^*\mathcal{O}). \quad \square$$

Theorem 2.3 implies in particular the following corollaries (see [13], Corollary 4.1 and Corollary 4.2).

Corollary 2.4. *Let $f : \mathcal{O}_1 \rightarrow \mathcal{O}'$ and $g : \mathcal{O}' \rightarrow \mathcal{O}_2$ be minimal holomorphic maps (resp. covering maps) between orbifolds. Then $g \circ f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ is a minimal holomorphic map (resp. covering map). \square*

Corollary 2.5. *Let $f : R_1 \rightarrow R'$ and $g : R' \rightarrow R_2$ be holomorphic branched covering maps, and $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$ orbifolds. Assume that $g \circ f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ is a minimal holomorphic map (resp. a covering map). Then $g : g^*\mathcal{O}_2 \rightarrow \mathcal{O}_2$ and $f : \mathcal{O}_1 \rightarrow g^*\mathcal{O}_2$ are minimal holomorphic maps (resp. covering maps). \square*

With each holomorphic function $f : R_1 \rightarrow R_2$ between compact Riemann surfaces one can associate in a natural way two orbifolds $\mathcal{O}_1^f = (R_1, \nu_1^f)$ and $\mathcal{O}_2^f = (R_2, \nu_2^f)$, setting $\nu_2^f(z)$ equal to the least common multiple of local degrees of f at the points of the preimage $f^{-1}\{z\}$, and

$$\nu_1^f(z) = \nu_2^f(f(z))/\deg_z f.$$

By construction, $f : \mathcal{O}_1^f \rightarrow \mathcal{O}_2^f$ is a covering map between orbifolds. For orbifolds \mathcal{O} and $\tilde{\mathcal{O}}$ write

$$\mathcal{O} \preceq \tilde{\mathcal{O}} \quad (21)$$

if for any $z \in \mathbb{CP}^1$ the condition $\nu(z) \mid \tilde{\nu}(z)$ holds. It is easy to see that the covering map $f : \mathcal{O}_1^f \rightarrow \mathcal{O}_2^f$ is minimal in the following sense. For any covering map between orbifolds $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ we have:

$$\mathcal{O}_1^f \preceq \mathcal{O}_1, \quad \mathcal{O}_2^f \preceq \mathcal{O}_2. \quad (22)$$

On the other hand, for any holomorphic map $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ we have:

$$f^* \mathcal{O}_2 \preceq \mathcal{O}_1.$$

Clearly, (21) implies that $\nu(\mathcal{O}) \leq \nu(\tilde{\mathcal{O}})$ but the inverse is not true in general.

Orbifolds turn out to be a convenient tool for the investigation of the functional equation

$$h = f \circ p = g \circ q, \quad (23)$$

where $p : R \rightarrow C_1$, $f : C_1 \rightarrow \mathbb{CP}^1$, $q : R \rightarrow C_2$, $g : C_2 \rightarrow \mathbb{CP}^1$, and $h : R \rightarrow \mathbb{CP}^1$ are holomorphic functions on compact Riemann surfaces. Denote by G_h the monodromy group G_h of the function h . Recall that decompositions of h into compositions

$$h = f \circ p \quad (24)$$

of holomorphic functions $p : R \rightarrow C_1$ and $f : C_1 \rightarrow \mathbb{CP}^1$, considered up to the change

$$\tilde{f} = f \circ \mu^{-1}, \quad \tilde{g} = \mu \circ g,$$

where $\mu : C_2 \rightarrow C_1$ is an isomorphism, are in a one-to-one correspondence with imprimitivity systems of G_h . We will call any holomorphic function p satisfying (24) for some holomorphic function f a *compositional right factor* of h . Compositional left factors of h are defined similarly.

We say that a solution h, f, p, g, q of (23) is *good* if any block of the imprimitivity systems of G_h corresponding to the decomposition $h = f \circ p$ intersects with any block of the imprimitivity systems of G_h corresponding to the decomposition $h = g \circ q$ and this intersection consists of a unique element. This is equivalent to the requirement that the fiber product of the coverings f and g has a unique component and p and q have no non-trivial common compositional right factor in the following sense: the equalities

$$p = \tilde{p} \circ w, \quad q = \tilde{q} \circ w, \quad (25)$$

where $w : R \rightarrow \tilde{R}$, $\tilde{p} : \tilde{R} \rightarrow C_1$, $\tilde{q} : \tilde{R} \rightarrow C_2$ are holomorphic functions, imply that $\deg w = 1$.

In the above notation the following statement holds ([13], Theorem 4.2).

Theorem 2.6. *Let h, f, p, g, q be a good solution of the equation $h = f \circ p = g \circ q$. Then the commutative diagram*

$$\begin{array}{ccc} \mathcal{O}_1^q & \xrightarrow{p} & \mathcal{O}_1^f \\ \downarrow q & & \downarrow f \\ \mathcal{O}_2^q & \xrightarrow{g} & \mathcal{O}_2^f \end{array} \quad (26)$$

consists of minimal holomorphic maps between orbifolds. \square

Notice that if f and g are rational functions, then the fiber product of f and g has a unique component if and only if the algebraic curve $f(x) - g(y) = 0$ is irreducible (see e.g. [12], Proposition 2.4). On the other hand, the Lüroth theorem implies that if p and q are rational functions, then they have no non-trivial common compositional right factor if and only if $\mathbb{C}(p, q) = \mathbb{C}(z)$. Notice also that the construction of the fiber product implies easily the following statement (see [13], Lemma 2.1).

Lemma 2.7. *A solution h, f, p, g, q of (23) is good whenever any two of the following three conditions are satisfied:*

- the fiber product of f and g has a unique component,
- p and q have no non-trivial common compositional right factor,
- $\deg f = \deg q, \quad \deg g = \deg p.$ □

Finally, let us mention the following more precise version of Proposition 2.1 for minimal holomorphic self-maps between orbifolds of positive characteristic (see [13], Theorem 5.1).

Theorem 2.8. *Let A and F be rational functions of degree at least two and \mathcal{O} an orbifold with $\chi(\mathcal{O}) > 0$ such that $A : \mathcal{O} \rightarrow \mathcal{O}$ is a holomorphic map between orbifolds and the diagram*

$$\begin{array}{ccc} \tilde{\mathcal{O}} & \xrightarrow{F} & \tilde{\mathcal{O}} \\ \downarrow \theta_{\mathcal{O}} & & \downarrow \theta_{\mathcal{O}} \\ \mathcal{O} & \xrightarrow{A} & \mathcal{O} \end{array} \quad (27)$$

is commutative. Then the following conditions are equivalent.

1. *The holomorphic map A is a minimal holomorphic map.*
2. *The homomorphism $\varphi : \Gamma_{\mathcal{O}} \rightarrow \Gamma_{\mathcal{O}}$ defined by the equality*

$$F \circ \sigma = \varphi(\sigma) \circ F, \quad \sigma \in \Gamma_{\mathcal{O}}, \quad (28)$$

is an automorphism of $\Gamma_{\mathcal{O}}$.

3. *The triple $F, A, \theta_{\mathcal{O}}$ is a good solution of the equation*

$$A \circ \theta_{\mathcal{O}} = \theta_{\mathcal{O}} \circ F. \quad \square$$

In this paper essentially all considered orbifolds will be defined on \mathbb{CP}^1 . The only exceptions from this rules are orbifolds which are universal coverings. So, usually we will omit the Riemann surface R in the definition of $\mathcal{O} = (R, \nu)$ meaning that $R = \mathbb{CP}^1$. We also will assume that all considered orbifolds are good that is have a universal covering.

3 Generalized Lattès maps

3.1 Equivalent descriptions

Say that a rational function A of degree at least two is a *generalized Lattès map* if there exists a good orbifold \mathcal{O} on \mathbb{CP}^1 distinct from the non-ramified sphere such that $A : \mathcal{O} \rightarrow \mathcal{O}$ is a minimal holomorphic map between orbifolds. Thus, we take as the basic definition the fourth condition of Theorem 1.1. The equivalence of this condition to the other conditions is shown in this subsection.

Notice that for a given rational function A there might be more than one orbifold \mathcal{O} such that $A : \mathcal{O} \rightarrow \mathcal{O}$ is a minimal holomorphic map and even infinite number of such orbifolds. Indeed, it is easy to see that $z^{\pm n} : \mathcal{O} \rightarrow \mathcal{O}$ is a minimal holomorphic map between orbifolds for any \mathcal{O} defined by the conditions

$$\nu(0) = \nu(\infty) = m, \quad \text{GCD}(n, m) = 1,$$

and $\pm T_n : \mathcal{O} \rightarrow \mathcal{O}$ is a minimal holomorphic map between orbifolds for any \mathcal{O} defined by the conditions

$$\nu(-1) = \nu(1) = 2, \quad \nu(\infty) = m, \quad \text{GCD}(n, m) = 1.$$

In the next section we will show however that if A is not conjugated to $z^{\pm d}$ or $\pm T_d$, then there exists an orbifold \mathcal{O}_0^A such that $A : \mathcal{O}_0^A \rightarrow \mathcal{O}_0^A$ is a minimal holomorphic map and for any orbifold \mathcal{O} such that $A : \mathcal{O} \rightarrow \mathcal{O}$ is a minimal holomorphic map the relation $\mathcal{O} \preceq \mathcal{O}_0^A$ holds.

Since $\deg A > 1$, it follows from (9) by Proposition 2.2 that $\chi(\mathcal{O}) \geq 0$ and $\chi(\mathcal{O}) = 0$ if and only if $A : \mathcal{O} \rightarrow \mathcal{O}$ is a covering map. Therefore, since a rational function f is a covering map $f : \mathcal{O} \rightarrow \mathcal{O}$ for some \mathcal{O} with $\chi(\mathcal{O}) = 0$ if and only if f is a Lattès map (see [10], Theorem 4.1), the class of generalized Lattès maps contains the class of classical Lattès maps. Further, it is well known and follows easily from (12) that the equality $\chi(\mathcal{O}) = 0$ holds if and only if $\nu(\mathcal{O})$ belongs to the list

$$\{2, 2, 2, 2\}, \quad \{3, 3, 3\}, \quad \{2, 4, 4\}, \quad \{2, 3, 6\}. \quad (29)$$

Similarly, for \mathcal{O} having a universal covering and distinct from the non-ramified sphere the inequality $\chi(\mathcal{O}) > 0$ holds if and only if $\nu(\mathcal{O})$ belongs to the list

$$\{n, n\}, \quad n \geq 2, \quad \{2, 2, n\}, \quad n \geq 2, \quad \{2, 3, 3\}, \quad \{2, 3, 4\}, \quad \{2, 3, 5\}. \quad (30)$$

Groups $\Gamma_{\mathcal{O}} \subset \text{Aut}(\mathbb{C})$ corresponding to orbifolds \mathcal{O} with ramification collections (29) are generated by translations of \mathbb{C} by elements of some lattice $L \subset \mathbb{C}$ of rank two and the transformation $z \rightarrow \varepsilon z$, where ε is an n th root of unity with n equal to 2, 3, 4, or 6, such that $\varepsilon L = L$ (see [10], or [6], Section IV.9.5). Accordingly, the functions $\theta_{\mathcal{O}}$ may be written in terms of the corresponding Weierstrass functions as $\wp(z)$, $\wp'(z)$, $\wp^2(z)$, and $\wp'^2(z)$. Groups $\Gamma_{\mathcal{O}} \subset \text{Aut}(\mathbb{CP}^1)$ corresponding to orbifolds \mathcal{O} with ramification collections (30) are C_n , D_{2n} , A_4 , S_4 , and A_5 , and the functions $\theta_{\mathcal{O}}$ are regular coverings of \mathbb{CP}^1 by \mathbb{CP}^1 of degrees n , $2n$, 12, 24, and 60, calculated for the first time by Klein in [7].

Let us prove the chain of implications $4 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1 \Rightarrow 4$ of the conditions of Theorem 1.1. By Proposition 2.1, if $A : \mathcal{O} \rightarrow \mathcal{O}$ is a minimal holomorphic map between orbifolds, then there exists a holomorphic map $B : \tilde{\mathcal{O}} \rightarrow \tilde{\mathcal{O}}$ and a homomorphism $\varphi : \Gamma_{\mathcal{O}} \rightarrow \Gamma_{\tilde{\mathcal{O}}}$ such that the diagram

$$\begin{array}{ccc} \tilde{\mathcal{O}} & \xrightarrow{B} & \tilde{\mathcal{O}} \\ \downarrow \theta_{\mathcal{O}} & & \downarrow \theta_{\mathcal{O}} \\ \mathcal{O} & \xrightarrow{A} & \mathcal{O} \end{array} \quad (31)$$

is commutative and

$$B \circ \sigma = \varphi(\sigma) \circ B, \quad \sigma \in \Gamma_{\mathcal{O}}, \quad (32)$$

for some homomorphism $\varphi : \Gamma_{\mathcal{O}} \rightarrow \Gamma_{\tilde{\mathcal{O}}}$. Furthermore, if $\chi(\mathcal{O}) > 0$ and $\tilde{\mathcal{O}} = \mathbb{CP}^1$, then the homomorphism φ is an automorphism by Theorem 2.8. If $\chi(\mathcal{O}) = 0$ and $\tilde{\mathcal{O}} = \mathbb{C}$, then φ is not necessarily surjective but remains a *monomorphism*. Indeed, in this case $A : \mathcal{O} \rightarrow \mathcal{O}$ is a covering map, implying that $B : \mathbb{C} \rightarrow \mathbb{C}$ is an isomorphism, that is has the form

$$B = az + b, \quad a, b \in \mathbb{C}. \quad (33)$$

Thus, B is invertible and hence the equality $B \circ \sigma = B$ implies that $\sigma = z$. This proves the implication $4 \Rightarrow 3$.

Since any monomorphism of a finite group is an automorphism, a proof of the implication $3 \Rightarrow 2$ is required only if $\chi(\mathcal{O}) = 0$ and $\tilde{\mathcal{O}} = \mathbb{C}$. Let $\Lambda_{\mathcal{O}}$ be the subgroup of $\Gamma_{\mathcal{O}}$ generated by translations. The subgroup $\Lambda_{\mathcal{O}}$ is normal in $\Gamma_{\mathcal{O}}$ and can be described as the kernel of the homomorphism $\psi : \Gamma_{\mathcal{O}} \rightarrow \mathbb{C}$ which sends $\sigma = \alpha z + \beta \in \Gamma_{\mathcal{O}}$ to $\psi(\sigma) = \alpha \in \mathbb{C}$. It is clear that $\mathcal{R} = \mathbb{C}/\Lambda_{\mathcal{O}}$ is a Riemann surface of genus one and that $\theta_{\mathcal{O}}$ can be decomposed as

$$\psi : \mathbb{C} \xrightarrow{\psi} \mathbb{C}/\Lambda_{\mathcal{O}} \cong \mathcal{R} \xrightarrow{\pi} \mathcal{R}/\Gamma \cong \mathbb{CP}^1,$$

where

$$\Gamma \cong \Gamma_{\mathcal{O}}/\Lambda_{\mathcal{O}} \subseteq \text{Aut}(\mathcal{R}).$$

Moreover, the classification of groups $\Gamma_{\mathcal{O}}$ given above implies that Γ is a cyclic group of order 2, 3, 4, or 6.

Since φ is a monomorphism, φ maps elements of infinite order of $\Gamma_{\mathcal{O}}$ to elements of infinite order. Therefore, $\varphi(\Lambda_{\mathcal{O}}) \subset \Lambda_{\tilde{\mathcal{O}}}$, implying that B descends to a holomorphic map $\tilde{B} : \mathcal{R} \rightarrow \mathcal{R}$ which makes the diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{B=ax+b} & \mathbb{C} \\ \downarrow \psi & & \downarrow \psi \\ \mathcal{R} & \xrightarrow{\tilde{B}} & \mathcal{R} \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array} \quad (34)$$

commutative. Finally, since B has form (33), it is easy to see that \tilde{B} commutes with Γ , that is the condition

$$\tilde{B} \circ \sigma = \tilde{\varphi}(\sigma) \circ \tilde{B}, \quad \sigma \in \Gamma,$$

holds for the identical automorphism $\tilde{\varphi}$. Notice that for $\nu(\mathcal{O}) = \{2, 2, 2, 2\}$ the complex structure of $\mathbb{C}/\Lambda_{\mathcal{O}}$ may be arbitrary, while for $\nu(\mathcal{O})$ equal $\{2, 4, 4\}$, $\{3, 3, 3\}$, or $\{2, 3, 6\}$ this structure is rigid and arises from the tiling of \mathbb{C} by squares, equilateral triangles, or alternately colored equilateral triangles, respectively.

In order to prove the implication $2 \Rightarrow 1$ it is enough to show that the maps π and B from diagram (5) have no non-trivial compositional right factor. If $\mathcal{R} = \mathbb{CP}^1$, this is a corollary of Theorem 2.8, so assume that the surface \mathcal{R} has genus one. In this case π is the quotient map $\pi : \mathcal{R} \rightarrow \mathcal{R}/\Gamma$ for a cyclic group Γ of order 2, 3, 4, or 6, implying that any compositional right factor of w has the form $w : \mathcal{R} \rightarrow \mathcal{R}/\tilde{\Gamma}$, where $\tilde{\Gamma}$ is a cyclic group. Therefore, w has a non-trivial ramification implying by the Riemann-Hurwitz formula that the surface $\mathcal{R}' = \mathcal{R}/\tilde{\Gamma}$ cannot have genus one. Thus, $g(\mathcal{R}') = 0$. On the other hand, for any decomposition

$$\mathcal{R} \xrightarrow{w} \mathcal{R}' \xrightarrow{\tilde{B}} \mathcal{R}$$

of B the inequality $g(\mathcal{R}) \leq g(\mathcal{R}') \leq g(\mathcal{R})$ holds, implying that $g(\mathcal{R}') = 1$. The contradiction obtained shows that B and π cannot have a non-trivial common compositional right factor.

Finally, the implication $1 \Rightarrow 4$ follows from Theorem 2.6. Indeed, this theorem implies that (9) holds for the orbifold \mathcal{O}_2^2 . On the other hand, since $\deg \pi \geq 2$, it follows from the definition of the orbifold \mathcal{O}_2^2 and the Riemann-Hurwitz formula that \mathcal{O}_2^2 cannot be the non-ramified sphere. This finishes the proof of Theorem 1.1.

Let us give an example illustrating Theorem 1.1. Let \mathcal{O} be an orbifold with ramification $\{2, 2, 2, 2\}$. Then $\tilde{\mathcal{O}} = \mathbb{C}$, the group $\Lambda_{\mathcal{O}}$ is generated by the transformations

$$\sigma_1 : z \rightarrow z + \omega_1, \quad \sigma_2 : z \rightarrow z + \omega_2,$$

for some $\omega_1, \omega_2 \in \mathbb{C}$ with $\omega_1/\omega_2 \notin \mathbb{R}$, and the group $\Gamma_{\mathcal{O}}$ is generated by $\Lambda_{\mathcal{O}}$ and the transformation $\delta : z \rightarrow -z$. An example of a function satisfying (32) is the function $B = mz$, where $m \geq 2$ is an integer. For this function (32) holds for the automorphism φ which maps the transformation δ to itself and the transformations σ_1, σ_2 to the transformations

$$z \rightarrow z + m\omega_1, \quad z \rightarrow z + m\omega_2.$$

Since $\theta_{\mathcal{O}} = \wp$, where $\wp = \wp_{\omega_1, \omega_2}$ is the Weierstrass function, equality (31) for the corresponding Lattès map $A = A_m$ reduces to the equality

$$\wp(mz) = A_m \circ \wp(z).$$

If the curve $\mathcal{R} = \mathbb{C}/\Lambda_{\mathcal{O}}$ is written in the Weierstrass form

$$y^2 = 4x^3 - g_2x - g_3, \quad g_2, g_3 \in \mathbb{C},$$

then the map π from the factorization $\wp = \pi \circ \psi$ is the x -projection of \mathcal{R} to \mathbb{CP}^1 . Notice that the homomorphism $\varphi : \Gamma_{\mathcal{O}} \rightarrow \Gamma_{\mathcal{O}}$ is not surjective. Notice also that for A_m there exist functions B and π satisfying (4) distinct from the ones given above. Namely, since for any $n, m \geq 2$ the functions A_n and A_m obviously commute, condition (4) holds for $\mathcal{R} = \mathbb{CP}^1$, $B = A_m$, and $\pi = A_n$. Furthermore, since $\deg A_m = m^2$, such B and π have no common compositional right factor whenever $\text{GCD}(n, m) = 1$.

3.2 Semiconjugacies and generalized Lattès functions

In this subsection we describe a general structure of solutions of (1) without the assumption that B and π have no non-trivial common compositional right factor. Recall that if A, B , and π are holomorphic maps satisfying (1), then the map B is called *semiconjugate* to the map A , and π is called a *semiconjugacy* from B to A . The semiconjugacy relation is not an equivalency relation. Nevertheless, it is easy to see that if B is semiconjugate to A , and A is semiconjugate to C , then B is semiconjugate to C .

For any decomposition $B = V \circ U$ of a rational function B into a composition of rational functions U and V the rational function $\tilde{B} = U \circ V$ is called an *elementary transformation* of B , and rational functions B and A are called *equivalent* if there exists a chain of elementary transformations

$$B \rightarrow B_1 \rightarrow B_2 \rightarrow \cdots \rightarrow B_s = A \quad (35)$$

between A and B . Clearly, chains (35) correspond to chains of equalities

$$U_i \circ V_i = V_{i+1} \circ U_{i+1}, \quad 1 \leq i \leq s-1, \quad (36)$$

where $U_i, V_i, 1 \leq i \leq s$, are rational functions, and

$$B = V_1 \circ U_1, \quad B_i = U_i \circ V_i, \quad 1 \leq i \leq s.$$

Notice that

$$\begin{aligned} B^{\circ s} &= (V_1 \circ U_1) \circ (V_1 \circ U_1) \circ \cdots \circ (V_1 \circ U_1) = V_1 \circ B_1^{\circ s-1} \circ U_1 = \\ &= V_1 \circ V_2 \circ B_2^{\circ i-2} \circ U_2 \circ U_1 = (V_1 \circ V_2 \circ \cdots \circ V_s) \circ (U_s \circ \cdots \circ U_2 \circ U_1). \end{aligned} \quad (37)$$

Since for any rational functions U, V the diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{B=V \circ U} & \mathbb{CP}^1 \\ \downarrow \pi=U & & \downarrow \pi=U \\ \mathbb{CP}^1 & \xrightarrow{A=U \circ V} & \mathbb{CP}^1 \end{array}$$

is commutative, the relation $A \sim B$ implies in particular that B is semiconjugate to A , and A is semiconjugate to B .

The notion of equivalence can be extended to self-maps between complex tori. Namely, if $B : \mathcal{R} \rightarrow \mathcal{R}$ is such a map, and $B = V \circ U$ is its decomposition into a composition of maps $V : \mathcal{R} \rightarrow \mathcal{R}'$ and $U : \mathcal{R}' \rightarrow \mathcal{R}$ between complex tori, we will call the map $U \circ V : \mathcal{R}' \rightarrow \mathcal{R}'$ an elementary transformation of B , and we will call maps B and A equivalent if there exists a chain of elementary transformations between B and A . Abusing the notation, below we will use for equivalent self-maps between complex tori the same symbol \sim .

Theorem 3.1. *Let \mathcal{R} be a compact Riemann surface of genus zero or one, and $A : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$, $B : \mathcal{R} \rightarrow \mathcal{R}$, and $\pi : \mathcal{R} \rightarrow \mathbb{CP}^1$ holomorphic maps such that diagram (1) is commutative. Then A is a generalized Lattès map, unless $\mathcal{R} = \mathbb{CP}^1$ and $B \sim A$. In more details, there exist a Riemann surface \mathcal{R}_0 of the same genus as \mathcal{R} and holomorphic maps $\psi : \mathcal{R} \rightarrow \mathcal{R}_0$, $\pi_0 : \mathcal{R}_0 \rightarrow \mathbb{CP}^1$, and $B_0 : \mathcal{R}_0 \rightarrow \mathcal{R}_0$ satisfying the following conditions.*

1. $B_0 \sim B$ and $\pi = \pi_0 \circ \psi$.

2. The diagram

$$\begin{array}{ccc}
 \mathcal{R} & \xrightarrow{B} & \mathcal{R} \\
 \downarrow \psi & & \downarrow \psi \\
 \mathcal{R}_0 & \xrightarrow{B_0} & \mathcal{R}_0 \\
 \downarrow \pi_0 & & \downarrow \pi_0 \\
 \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1
 \end{array} \tag{38}$$

is commutative.

3. The maps π_0 and B_0 have no non-trivial common compositional right factor. The map π_0 has degree at least two, unless $\mathcal{R} = \mathbb{CP}^1$ and $B \sim A$.

4. The maps $A : \mathcal{O}_2^{\pi_0} \rightarrow \mathcal{O}_2^{\pi_0}$ and $B_0 : \mathcal{O}_1^{\pi_0} \rightarrow \mathcal{O}_1^{\pi_0}$ are minimal holomorphic maps between orbifolds.

5. The map ψ is a compositional right factor of some iteration $B^{\circ k}$, $k \geq 1$.

Proof. If the collection

$$f = A, \quad p = \pi, \quad g = \pi, \quad q = B \tag{39}$$

is a good solution of (23), we can set $B_0 = B$, $\pi_0 = \pi$. Then $A : \mathcal{O}_2^{\pi_0} \rightarrow \mathcal{O}_2^{\pi_0}$ and $B_0 : \mathcal{O}_1^{\pi_0} \rightarrow \mathcal{O}_1^{\pi_0}$ are minimal holomorphic maps by Theorem 2.6. Further, if $\deg \pi_0 = 1$, then A and B conjugate implying that $A \sim B$. Therefore, unless $\mathcal{R} = \mathbb{CP}^1$ and $B \sim A$, the map π_0 has degree at least two, implying that A is a generalized Lattès map. The other conditions hold trivially.

Assume now that (39) is not a good solution of (23). Since

$$\deg p = \deg g = \deg \pi,$$

by Lemma 2.7 this is equivalent to the condition that π and B have a non-trivial common compositional right factor. Thus, there exist a Riemann surface \mathcal{R}' and holomorphic maps

$$U_1 : \mathcal{R} \rightarrow \mathcal{R}', \quad \pi' : \mathcal{R}' \rightarrow \mathbb{CP}^1, \quad V_1 : \mathcal{R}' \rightarrow \mathcal{R},$$

such that

$$\pi = \pi' \circ U_1, \quad B = V_1 \circ U_1, \quad (40)$$

and $\deg U_1 \geq 2$. Furthermore, since $B : \mathcal{R} \rightarrow \mathcal{R}$ is decomposed as

$$\mathcal{R} \xrightarrow{U_1} \mathcal{R}' \xrightarrow{V_1} \mathcal{R},$$

the equality $g(\mathcal{R}') = g(\mathcal{R})$ holds.

Clearly, equalities (40) imply that the diagram

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{B} & \mathcal{R} \\ \downarrow U_1 & & \downarrow U_1 \\ \mathcal{R}' & \xrightarrow{U_1 \circ V_1} & \mathcal{R}' \\ \downarrow \pi' & & \downarrow \pi' \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array}$$

is commutative. If the solution

$$f = A, \quad p = \pi', \quad g = \pi', \quad q = U_1 \circ V_1$$

of (23) is still not good, we can perform a similar transformation once again. Since $\deg U_1 \geq 2$ implies that $\deg \pi' < \deg \pi$, it is clear that after a finite number of steps we will arrive to diagram (38), where B_0 is obtained from B by the chain of elementary transformations (36),

$$\psi = U_s \circ \cdots \circ U_2 \circ U_1,$$

and

$$f = A, \quad p = \pi_0, \quad g = \pi_0, \quad q = B_0, \quad (41)$$

is a good solution of (39). Furthermore, $\deg \pi_0 = 1$ only if $\mathcal{R} = \mathbb{CP}^1$ and $B \sim A$. Applying now as above Theorem 2.6 to (41) we conclude that $A : \mathcal{O}_2^{\pi_0} \rightarrow \mathcal{O}_2^{\pi_0}$ and $B_0 : \mathcal{O}_1^{\pi_0} \rightarrow \mathcal{O}_1^{\pi_0}$ are minimal holomorphic maps between orbifolds. Finally, ψ is a compositional factor of some B^{os} , $s \geq 1$, by (37). \square

Notice that if A is a generalized Lattès map, then the function π_0 from Theorem 3.1 is related to the quotient map π from diagram (7). Indeed, since

$B_0 : \mathcal{O}_1^{\pi_0} \rightarrow \mathcal{O}_1^{\pi_0}$ is a minimal holomorphic map, it follows from Proposition 2.1 that we can complete the low square of diagram (38) to the diagram

$$\begin{array}{ccc} \widetilde{\mathcal{R}}_0 & \xrightarrow{\widetilde{B}} & \widetilde{\mathcal{R}}_0 \\ \downarrow \widetilde{\psi} & & \downarrow \widetilde{\psi} \\ \mathcal{R}_0 & \xrightarrow{B_0} & \mathcal{R}_0 \\ \downarrow \pi_0 & & \downarrow \pi_0 \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1, \end{array}$$

where $\widetilde{\psi}$ is a universal covering of $\mathcal{O}_1^{\pi_0}$, and Corollary 2.4 implies that $\pi_0 \circ \widetilde{\psi}$ is a universal cover of $\mathcal{O}_2^{\pi_0}$. Furthermore, if $\deg \pi_0 \geq 2$, then the Riemann-Hurwitz formula implies that $\mathcal{O}_2^{\pi_0}$ is distinct from the non-ramified sphere. Thus, π_0 is a *compositional left factor* of some quotient map π in (7).

Theorem 3.1 shows that the problem of describing of rational solutions of the functional equation

$$A \circ \pi = \pi \circ B \quad (42)$$

“up to equivalence” reduces to the case where $\chi(\mathcal{O}_2^\pi) \geq 0$ ([13]). Moreover, it was shown in the paper [15] basing on methods of [13] that for any good rational solution of the more general functional equation

$$A \circ \delta = \pi \circ B, \quad (43)$$

such that

$$\deg A \geq 84 \deg \pi$$

the inequality $\chi(\mathcal{O}_2^\pi) \geq 0$ still holds. The rational functions π with $\chi(\mathcal{O}_2^\pi) \geq 0$ can be characterized by the condition that the genus of the Galois closure of $\mathbb{C}(z)/\mathbb{C}(\pi)$ equals zero or one (see [15]). In other words, these functions are compositional left factors of Galois coverings $\pi : \mathcal{R} \rightarrow \mathbb{CP}^1$, where \mathcal{R} is a compact Riemann surface of genus zero or one. This implies in particular that up to the change

$$A \rightarrow \mu_1 \circ A \circ \mu_2,$$

where μ_1 and μ_2 are Möbius transformations, besides the functions

$$z^n, \quad T_n, \quad \frac{1}{2} \left(z^n + \frac{1}{z^n} \right), \quad n \geq 1, \quad (44)$$

the class of functions with $\chi(\mathcal{O}_2^\pi) > 0$ contains only a finite number of functions which can be calculated explicitly. For a detailed description of such functions we refer the reader to the paper [16]. Notice that the problem of describing of rational solutions of functional equations (42) and (43) naturally arises in arithmetics and dynamics (see e. g. [1], [5], [9], [14]).

3.3 Compositions and decompositions

For a given orbifold \mathcal{O} with $\chi(\mathcal{O}) \geq 0$ denote by $\mathcal{E}(\mathcal{O})$ the set of rational functions A such that $A : \mathcal{O} \rightarrow \mathcal{O}$ is a minimal holomorphic map.

Theorem 3.2. *For any rational functions U and V contained in $\mathcal{E}(\mathcal{O})$ the composition $U \circ V$ also is contained in $\mathcal{E}(\mathcal{O})$. In the other direction, if U and V are rational functions such that the composition $U \circ V$ is contained in $\mathcal{E}(\mathcal{O})$ and $\nu(\mathcal{O}) \neq \{2, 2, 2, 2\}$, then there exists a Möbius transformation μ such that $U \circ \mu$ and $\mu^{-1} \circ V$ are contained in $\mathcal{E}(\mathcal{O})$. If $\nu(\mathcal{O}) = \{2, 2, 2, 2\}$, then we only may state that there exists an orbifold \mathcal{O}' such that $\nu(\mathcal{O}') = \{2, 2, 2, 2\}$ and $V : \mathcal{O} \rightarrow \mathcal{O}'$ and $U : \mathcal{O}' \rightarrow \mathcal{O}$ are minimal holomorphic maps.*

Proof. If U, V are contained in $\mathcal{E}(\mathcal{O})$, then Corollary 2.4 obviously implies that the composition $U \circ V$ also is contained in $\mathcal{E}(\mathcal{O})$.

In the other direction, assume that $U \circ V \in \mathcal{E}(\mathcal{O})$, and set $\mathcal{O}' = U^*\mathcal{O}$. Since by Theorem 2.3 the maps $U : \mathcal{O}' \rightarrow \mathcal{O}$ and $V : \mathcal{O} \rightarrow \mathcal{O}'$ are minimal holomorphic maps between orbifolds, we have:

$$\nu(\mathcal{O}) \leq \nu(\mathcal{O}') \leq \nu(\mathcal{O}). \quad (45)$$

Furthermore, by Proposition 2.2, we have:

$$\chi(\mathcal{O}) \leq \chi(\mathcal{O}') \deg V, \quad \chi(\mathcal{O}') \leq \chi(\mathcal{O}) \deg U.$$

Therefore,

$$\chi(\mathcal{O}) \leq \chi(\mathcal{O}') \deg V \leq \chi(\mathcal{O}) \deg U \deg V,$$

implying that $\chi(\mathcal{O}') = 0$ whenever $\chi(\mathcal{O}) = 0$, and $\chi(\mathcal{O}') > 0$ whenever $\chi(\mathcal{O}) > 0$.

Assume first that $\chi(\mathcal{O}) = 0$. A direct analysis of the table

Table 1

	$\{2, 2, 2, 2\}$	$\{3, 3, 3\}$	$\{2, 4, 4\}$	$\{2, 3, 6\}$
$\{2, 2, 2, 2\}$	\leq		\leq	\leq
$\{3, 3, 3\}$		\leq		\leq
$\{2, 4, 4\}$			\leq	
$\{2, 3, 6\}$				\leq

listing all $\nu(\mathcal{O}_1)$ and $\nu(\mathcal{O}_2)$ such that $\chi(\mathcal{O}_1) = \chi(\mathcal{O}_2) = 0$ and $\nu(\mathcal{O}_1) \leq \nu(\mathcal{O}_2)$, shows that (45) is possible only if $\nu(\mathcal{O}') = \nu(\mathcal{O})$. Finally, if $\nu(\mathcal{O}) \neq \{2, 2, 2, 2\}$, the orbifold \mathcal{O} has three ramification points, implying that we can find a Möbius transformation μ as required.

If $\chi(\mathcal{O}) > 0$ the proof can be done as follows (cf. [13], Corollary 5.1). Since $U : \mathcal{O}' \rightarrow \mathcal{O}$ and $V : \mathcal{O} \rightarrow \mathcal{O}'$ are minimal holomorphic maps, it follows from Proposition 2.1 that there exist rational functions F_U and F_V which make the

diagram

$$\begin{array}{ccccc}
\mathbb{CP}^1 & \xrightarrow{F_V} & \mathbb{CP}^1 & \xrightarrow{F_U} & \mathbb{CP}^1 \\
\downarrow \theta_{\mathcal{O}} & & \downarrow \theta_{\mathcal{O}'} & & \downarrow \theta_{\mathcal{O}} \\
\mathcal{O} & \xrightarrow{V} & \mathcal{O}' & \xrightarrow{U} & \mathcal{O}
\end{array}$$

commutative and satisfy

$$F_V \circ \sigma = \varphi_V(\sigma) \circ F_V, \quad \sigma \in \Gamma_{\mathcal{O}}, \quad F_U \circ \sigma = \varphi_U(\sigma) \circ F_U, \quad \sigma \in \Gamma_{\mathcal{O}'}$$

for some homomorphisms

$$\varphi_V : \Gamma_{\mathcal{O}} \rightarrow \Gamma_{\mathcal{O}'}, \quad \varphi_U : \Gamma_{\mathcal{O}'} \rightarrow \Gamma_{\mathcal{O}}.$$

Since the function $F_U \circ F_V$ makes the diagram

$$\begin{array}{ccc}
\mathbb{CP}^1 & \xrightarrow{F_V \circ F_U} & \mathbb{CP}^1 \\
\downarrow \theta_{\mathcal{O}} & & \downarrow \theta_{\mathcal{O}} \\
\mathcal{O} & \xrightarrow{V \circ U} & \mathcal{O}
\end{array}$$

commutative, Theorem 2.8 implies that the composition of homomorphisms

$$\varphi_U \circ \varphi_V : \Gamma_{\mathcal{O}} \rightarrow \Gamma_{\mathcal{O}}$$

is an automorphism. Therefore, $\Gamma_{\mathcal{O}'} \cong \Gamma_{\mathcal{O}}$, implying that $\nu(\mathcal{O}') = \nu(\mathcal{O})$. \square

Theorem 3.2, which seems to be new even for classical Lattès maps, in a sense reduces the study of generalized Lattès maps corresponding to orbifolds \mathcal{O} with $\nu(\mathcal{O}) \neq \{2, 2, 2, 2\}$ to the study of indecomposable maps. More precisely, the following statement holds.

Corollary 3.3. *Any rational function A contained in $\mathcal{E}(\mathcal{O})$ has a maximal decomposition*

$$A = A_1 \circ A_2 \circ \cdots \circ A_l$$

whose elements are contained in $\mathcal{E}(\mathcal{O})$. \square

Proof. Indeed, if A is indecomposable we have nothing to prove. Otherwise, $A = U \circ V$ for some rational functions U and V , and changing U to $U \circ \mu$ and V to $\mu^{-1} \circ V$, where μ a Möbius transformation provided by Theorem 3.2, without loss of generality we may assume that $U, V \in \mathcal{E}(\mathcal{O})$. Continuing in this way we will obtain the required maximal decomposition. \square

Corollary 3.4. *Assume that $A \in \mathcal{E}(\mathcal{O})$ and $B \sim A$. Then B is conjugate to some $B' \in \mathcal{E}(\mathcal{O})$.*

Proof. By Theorem 3.2, the statement of the corollary is true if B is a elementary transformation A . Therefore, it is true for any $B \sim A$. \square

4 Orbifold \mathcal{O}_0^A and symmetries

4.1 Extended groups of symmetries

For a rational function F define $\mathcal{G}(F)$ as a collection of Möbius transformations σ such that

$$F \circ \sigma = \nu_\sigma \circ F \quad (46)$$

for some Möbius transformations ν_σ . It is easy to see that in fact $\mathcal{G}(F)$ is a group with respect to the composition operation, and the map

$$\gamma_F : \sigma \rightarrow \nu_\sigma$$

is a homomorphism from $\mathcal{G}(F)$ to the group $\text{Aut}(\mathbb{CP}^1)$. We will call $\mathcal{G}(F)$ the *extended group of symmetries*. Abusing the notation, we will keep using the symbol \circ for the group operation in $\text{Aut}(\mathbb{CP}^1)$.

In this section we show that, unless

$$F = \mu_1 \circ z^d \circ \mu_2$$

for some $\mu_1, \mu_2 \in \text{Aut}(\mathbb{CP}^1)$, the group $\mathcal{G}(F)$ is finite and the order of any element of $\mathcal{G}(F)$ is bounded from above by $\deg F$. Using this fact, we show that for any minimal holomorphic map between orbifolds $A : \mathcal{O} \rightarrow \mathcal{O}$ with $\nu(\mathcal{O}) = \{n, n\}$ or $\nu(\mathcal{O}) = \{2, 2, n\}$ the inequality $n \leq \deg A$ holds, unless A is conjugated to $z^{\pm d}$ or $\pm T_d$. Finally, we show that for any rational function A there exists an orbifold \mathcal{O}_0^A such that $A : \mathcal{O}_0^A \rightarrow \mathcal{O}_0^A$ is a minimal holomorphic map and for any orbifold \mathcal{O} such that $A : \mathcal{O} \rightarrow \mathcal{O}$ is a minimal holomorphic map the relation $\mathcal{O} \preceq \mathcal{O}_0^A$ holds.

Since for any $\mu_1, \mu_2 \in \text{Aut}(\mathbb{CP}^1)$ equality (46) implies the equality

$$(\mu_1 \circ F \circ \mu_2) \circ (\mu_2^{-1} \circ \sigma \circ \mu_2) = (\mu_1 \circ \nu_\sigma \circ \mu_1^{-1}) \circ (\mu_1 \circ F \circ \mu_2),$$

we have:

$$\mathcal{G}(\mu_1 \circ F \circ \mu_2) = \mu_2^{-1} \circ \mathcal{G}(F) \circ \mu_2. \quad (47)$$

In particular,

$$\mathcal{G}(\mu \circ F) = \mathcal{G}(F)$$

for any $\mu \in \text{Aut}(\mathbb{CP}^1)$. On the other hand, for any subgroup Γ of $\mathcal{G}(F)$ and $\mu \in \text{Aut}(\mathbb{CP}^1)$, we have:

$$\gamma_{\mu \circ F}(\Gamma) = \mu \circ \gamma_F(\Gamma) \circ \mu^{-1}. \quad (48)$$

In order to shorten the notation, we will use an equivalence relation \sim_μ defined on the set of rational functions as follows:

$$A_1 \sim_\mu A_2,$$

if A_1 and A_2 satisfy the equality

$$A_1 = \mu_1 \circ A_2 \circ \mu_2$$

for some $\mu_1, \mu_2 \in \text{Aut}(\mathbb{CP}^1)$ (we use the subscript μ in order to distinguish this relation with the relation \sim introduced above).

Lemma 4.1. *Let $F = z^d$. Then the group $\mathcal{G}(F)$ consists of the transformations $\sigma = cz^{\pm 1}$, $c \in \mathbb{C}$.*

Proof. It is clear that any $\sigma = cz^{\pm 1}$, $c \in \mathbb{C}$, is contained in $\mathcal{G}(F)$. On the other hand, since for any $\sigma \in \text{Aut}(\mathbb{CP}^1)$ the preimage $(F \circ \sigma)^{-1}\{0, \infty\}$ consists of two points, equality (46) implies that $\nu_\sigma^{-1}\{0, \infty\} = \{0, \infty\}$. Therefore, for any $\sigma \in \mathcal{G}(F)$ we have $\nu_\sigma = bz^{\pm 1}$, $b \in \mathbb{C}$, implying easily that σ has the required form. \square

The situation described in Lemma 4.1 is exceptional as the following statement shows.

Theorem 4.2. *Let F be a rational function of degree $d \geq 2$ such that $F \not\sim_\mu z^d$. Then the group $\mathcal{G}(F)$ is one of the five finite rotation groups of the sphere $A_4, S_4, A_5, C_n, D_{2n}$, and the order of any element of $\mathcal{G}(F)$ does not exceed d . In particular, $|\mathcal{G}(F)| \leq \max\{60, 2d\}$.*

Proof. Any non-identical element of the group $\text{Aut}(\mathbb{CP}^1) \cong \text{PSL}_2(\mathbb{C})$ is conjugated either to $z \rightarrow z + 1$ or to $z \rightarrow \lambda z$ for some $\lambda \in \mathbb{C} \setminus \{0, 1\}$. Thus, making the change

$$F \rightarrow \mu_1 \circ F \circ \mu_2, \quad \sigma \rightarrow \mu_2^{-1} \circ \sigma \circ \mu_2, \quad \nu_\sigma \rightarrow \mu_1 \circ \nu_\sigma \circ \mu_1^{-1} \quad (49)$$

for convenient $\mu_1, \mu_2 \in \text{Aut}(\mathbb{CP}^1)$, without loss of generality we may assume that σ and ν_σ in (46) have one of the two forms above. Since any of the equalities

$$F(z + 1) = \lambda F(z), \quad \lambda \in \mathbb{C} \setminus \{0, 1\},$$

or

$$F(z + 1) = F(z) + 1$$

implies that $F(z)$ has infinitely many poles, they are impossible. Furthermore, considering the Laurent series at infinity of both parts of the equality

$$F(\lambda z) = F(z) + 1, \quad \lambda \in \mathbb{C} \setminus \{0, 1\}, \quad (50)$$

we conclude that this equality is impossible either. Thus,

$$F(\lambda_1 z) = \lambda_2 F(z), \quad \lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0, 1\}. \quad (51)$$

In particular, the group $\mathcal{G}(F)$ and its image under γ_F in $\text{Aut}(\mathbb{CP}^1)$ are rotations groups.

Considering the Laurent series at infinity of both parts of (51) and taking into account that these series contain more than one term since $F \not\sim_\mu z^d$, we conclude that λ_1 is a root of unity and $F = z^r R(z^n)$, where $R \in \mathbb{C}(z)$, $r \geq 0$,

and $n = \text{ord}(\lambda_1) = \text{ord}(\mu)$. In particular, this implies that for any $\mu \in \mathcal{G}(F)$ the inequality

$$\text{ord}(\mu) \leq d \quad (52)$$

holds. Indeed, since $F \not\sim_{\mu} z^d$, the preimage $F^{-1}\{0, \infty\}$ contains a point $p \neq 0, \infty$.

For such p we have:

$$d \geq \deg_p F \geq n \deg_{p^n} R \geq n.$$

Show now that $\mathcal{G}(F)$ is finite. Assume the inverse, and let $\sigma_1, \sigma_2, \dots, \sigma_s, \dots$ be an infinite sequence of pairwise distinct elements of $\mathcal{G}(F)$. Observe first that for any $s \geq 1$ the group

$$\Gamma_s = \langle \sigma_1, \sigma_2, \dots, \sigma_s \rangle$$

is finite. Indeed, if Γ_s is infinite, then the lifting $\overline{\Gamma}_s$ of Γ_s to $\text{SL}_2(\mathbb{C})$ also is infinite. Since by the Schur Theorem (see e.g. [2], (36.2)) any infinite finitely generated subgroup of $\text{GL}_k(\mathbb{C})$ has an element of infinite order, this implies that $\overline{\Gamma}_s$ has an element of infinite order. But in this case Γ_s also has an element of infinite order in contradiction with (52).

Further, since elements $\sigma_1, \sigma_2, \dots, \sigma_s, \dots$ of the group $\mathcal{G}(F)$ are distinct, $|\Gamma_s| \rightarrow \infty$. On the other hand, since the groups Γ_s , $s \geq 1$, are finite rotations groups, they belong to the list $A_4, S_4, A_5, C_n, D_{2n}$. Therefore, $|\Gamma_s| \rightarrow \infty$ yields that for s big enough the group Γ_s is either C_n or D_{2n} with $n > d$. However, since both groups C_n and D_{2n} have an element of order n , this contradicts to (52). Therefore, $\mathcal{G}(F)$ is finite. Finally, if $\mathcal{G}(F)$ is A_4, S_4 , or A_5 , then $\mathcal{G}(F) \leq 60$, while if $\mathcal{G}(F)$ is C_n or D_{2n} , then $n \leq d$, since C_n and D_{2n} have an element of order n . \square

Theorem 4.3. *Let A be a rational functions of degree $d \geq 2$ such that $A : \mathcal{O} \rightarrow \mathcal{O}$ is a minimal holomorphic map between orbifolds with $\nu(\mathcal{O}) = \{n, n\}$, $n \geq 2$, or $\nu(\mathcal{O}) = \{2, 2, n\}$, $n \geq 2$. Then the inequality $n \leq d$ holds, unless A is conjugated to $z^{\pm d}$ or $\pm T_d$ for some d coprime with n .*

Proof. By Theorem 2.8, the rational function A lifts to a rational function F which makes diagram (27) commutative and satisfies (28) for some automorphism $\varphi : \Gamma_{\mathcal{O}} \rightarrow \Gamma_{\mathcal{O}}$. Since this implies that the group $\Gamma_{\mathcal{O}}$, which is either C_n or D_{2n} , is a subgroup of $\mathcal{G}(F)$ and φ is a restriction of γ_F , it follows from the equality $\deg F = d$ and Theorem 4.2 that $n \leq d$, unless $F \not\sim_{\mu} z^d$. Thus, we only

must consider the case $F \sim_{\mu} z^d$.

Assume first that $\nu(\mathcal{O}) = \{n, n\}$, $n \geq 2$. Making the change

$$A \rightarrow \mu_1 \circ A \circ \mu_1^{-1}, \quad X \rightarrow \mu_1 \circ X$$

for convenient $\mu_1 \in \text{Aut}(\mathbb{CP}^1)$, without loss of generality we may assume that

$$\nu(0) = n, \quad \nu(\infty) = n.$$

Then $\theta_{\mathcal{O}} = z^n \circ \mu_2$, where $\mu_2 \in \text{Aut}(\mathbb{CP}^1)$, and making the change

$$X \rightarrow X \circ \mu_2, \quad \nu_{\mu} \rightarrow \mu_2^{-1} \circ F \circ \mu_2,$$

we can assume that $\theta_{\mathcal{O}} = z^n$ and the group $\Gamma_{\mathcal{O}} = C_n$ is generated by the transformation

$$\alpha : z \rightarrow e^{2\pi i/n} z.$$

Since $\alpha \in \mathcal{G}(F)$ and $F = \delta_1 \circ z^d \circ \delta_2$ for some Möbius transformations δ_1 and δ_2 , it follows from Lemma 4.1 that $\delta_2 \circ \alpha$ has the form $cz^{\pm 1}$, $c \in \mathbb{C}$. Therefore, δ_2 also has such a form and hence

$$F = \mu \circ z^d, \quad \mu \in \text{Aut}(\mathbb{CP}^1). \quad (53)$$

Furthermore, since φ is a restriction of γ_F and for any automorphism $\varphi : \Gamma_{\mathcal{O}} \rightarrow \Gamma_{\mathcal{O}}$ we have

$$\varphi(\alpha) = \alpha^{\circ r}, \quad 1 \leq r \leq n-1, \quad \text{GCD}(n, r) = 1, \quad (54)$$

it follows from (46) that

$$\mu \circ \alpha^{\circ d} = \alpha^{\circ r} \circ \mu \quad (55)$$

for some r as above. Since (55) implies that $\mu^{-1}\{0, \infty\} = \{0, \infty\}$, the equality $\mu = cz^{\pm 1}$, $c \in \mathbb{C}$ holds, and hence

$$F = cz^{\pm d}, \quad c \in \mathbb{C}. \quad (56)$$

Moreover, (55) implies that $\text{GCD}(d, n) = 1$. It follows now from the equality

$$A \circ \theta_{\mathcal{O}} = \theta_{\mathcal{O}} \circ F \quad (57)$$

that $A = c^n z^{\pm d}$, implying that A is conjugated to $z^{\pm d}$.

Assume now that $\nu(\mathcal{O}) = \{2, 2, n\}$, $n \geq 2$. Since for $n = 2$ the inequality $n \leq d$ obviously holds, we may assume that $n > 2$. Furthermore, we may assume that

$$\nu(-1) = 2, \quad \nu(1) = 2, \quad \nu(\infty) = n,$$

the group $\Gamma_{\mathcal{O}} = D_n$ is generated by the transformations

$$\alpha : z \rightarrow e^{2\pi i/n} z, \quad \beta : z \rightarrow \frac{1}{z},$$

and

$$\theta_{\mathcal{O}} = \frac{1}{2} \left(z^n + \frac{1}{z^n} \right). \quad (58)$$

As above, it follows from $\alpha \in \mathcal{G}(F)$ that $F = \mu \circ z^d$ for some $\mu \in \text{Aut}(\mathbb{CP}^1)$. Furthermore, since $\Gamma_{\mathcal{O}} = D_{2n}$ with $n > 2$ and the automorphism φ maps any element of order n of $\Gamma_{\mathcal{O}}$ to an element of order n , equality (54) still holds, implying as above that equality (56) holds. In particular,

$$\varphi(\beta) = c^2 \circ \frac{1}{z}. \quad (59)$$

On the other hand, since φ maps any element of order two of Γ_Θ to an element of order two not belonging to the subgroup generated by α , we have:

$$\varphi(\beta) = \alpha^{\circ k} \circ \beta = e^{2\pi i k/n} z \circ \frac{1}{z}, \quad 0 \leq k \leq n-1. \quad (60)$$

Since (59) and (60) yield that $c^{2n} = 1$, equalities (58) and (57) imply that

$$A \circ \frac{1}{2} \left(z^n + \frac{1}{z^n} \right) = \pm \frac{1}{2} \left(z^{dn} + \frac{1}{z^{dn}} \right). \quad (61)$$

It follows now from (61) and the well-known identity

$$T_d \circ \frac{1}{2} \left(z^n + \frac{1}{z^n} \right) = \frac{1}{2} \left(z^n + \frac{1}{z^n} \right) \circ z^d = \frac{1}{2} \left(z^{dn} + \frac{1}{z^{dn}} \right),$$

that $A = \pm T_d$. □

4.2 Orbifold \mathcal{O}_0^A

For orbifolds $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_s$ define the orbifold $\mathcal{O} = \text{LCM}(\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_s)$ by the condition

$$\nu(z) = \text{LCM}(\nu_1(z), \nu_2(z), \dots, \nu_s(z)), \quad z \in \mathbb{CP}^1.$$

Proposition 4.4. *Let $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_s$ and $\mathcal{O}'_1, \mathcal{O}'_2, \dots, \mathcal{O}'_s$ be orbifolds, and A a rational function such that the maps $A : \mathcal{O}_i \rightarrow \mathcal{O}'_i$, $1 \leq i \leq s$, are holomorphic maps (resp. minimal holomorphic maps, covering maps) between orbifolds. Then*

$$A : \text{LCM}(\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_s) \rightarrow \text{LCM}(\mathcal{O}'_1, \mathcal{O}'_2, \dots, \mathcal{O}'_s)$$

also is a holomorphic map (resp. a minimal holomorphic map, a covering map) between orbifolds.

Proof. In order to prove the first part of the proposition, it is enough to observe that the conditions

$$\nu'_i(A(z)) \mid \nu_i(z) \deg_z A, \quad 1 \leq i \leq s,$$

imply the condition

$$\begin{aligned} & \text{LCM}(\nu'_1(A(z)), \nu'_2(A(z)), \dots, \nu'_s(A(z))) \mid \\ & \text{LCM}(\nu_1(z) \deg_z A, \nu_2(z) \deg_z A, \dots, \nu_s(z) \deg_z A) = \\ & \text{LCM}(\nu_1(z), \nu_2(z), \dots, \nu_s(z)) \deg_z A. \end{aligned}$$

In order to prove the second part, we must show that if

$$\nu'_i(A(z)) = \nu_i(z) \text{GCD}(\nu'_i(A(z)), \deg_z A), \quad 1 \leq i \leq s,$$

then

$$\begin{aligned} \text{LCM}\left(\nu'_1(A(z)), \nu'_2(A(z)), \dots, \nu'_s(A(z))\right) &= \text{LCM}\left(\nu_1(z), \nu_2(z), \dots, \nu_s(z)\right) \times \\ &\times \text{GCD}\left(\text{LCM}\left(\nu'_1(A(z)), \nu'_2(A(z)), \dots, \nu'_s(A(z))\right), \deg_z A\right). \end{aligned} \quad (62)$$

Let p be an arbitrary prime number and $z \in \mathbb{CP}^1$. Set

$$b_i = \text{ord}_p \nu'_i(A(z)), \quad a_i = \text{ord}_p \nu_i(z), \quad c = \text{ord}_p \deg_z A, \quad 1 \leq i \leq s.$$

Considering the orders at p of the both parts of equality (62), we see that the proof of the theorem reduces to the proof of the following statement: if a_i , b_i , $1 \leq i \leq s$, and c are integer non-negative numbers such that

$$b_i = a_i + \min\{c, b_i\}, \quad 1 \leq i \leq s, \quad (63)$$

then

$$\max_i \{b_i\} = \max_i \{a_i\} + \min\{c, \max_i \{b_i\}\}. \quad (64)$$

For each i , $1 \leq i \leq s$, equality (63) means that $b_i = a_i + c$, if $c \leq b_i$, and $a_i = 0$, if $c > b_i$. Let I_1 (resp. I_2) be the subset of $\{1, 2, \dots, s\}$ consisting of indices i such that $b_i \geq c$ (resp. $b_i < c$). Clearly, we have:

$$\max_i \{b_i\} = \max \left\{ \max_{i \in I_1} \{b_i\}, \max_{i \in I_2} \{b_i\} \right\}$$

If $c > \max_i \{b_i\}$, that is the set I_1 is empty, then $\max_i \{a_i\} = 0$, and hence (64) holds.

If $c \leq \max_i \{b_i\}$, then for at least one i_0 , $1 \leq i_0 \leq s$, we have $c \leq b_{i_0}$ and $b_{i_0} = a_{i_0} + c$, implying that for any $i \in I_2$ we have:

$$b_i < c \leq c + a_{i_0} = b_{i_0} \leq \max_{i \in I_1} \{b_i\}.$$

Thus,

$$\max_i \{b_i\} = \max_{i \in I_1} \{b_i\} = \max_{i \in I_1} \{a_i + c\} = \max_{i \in I_1} \{a_i\} + c.$$

Furthermore, since $a_i = 0$ whenever $i \in I_2$, we have:

$$\max_{i \in I_1} \{a_i\} = \max_i \{a_i\}.$$

Therefore, if $c \leq \max_i \{b_i\}$ equality (64) also holds.

Finally, since a minimal holomorphic map $f : \mathcal{O} \rightarrow \mathcal{O}'$ is a covering map if and only if $\deg_z A \mid \nu'_i(A(z))$ for any $z \in \mathbb{CP}^1$, in order to prove the third part it is enough to observe that the conditions

$$\deg_z A \mid \nu'_i(A(z)), \quad 1 \leq i \leq s, \quad z \in \mathbb{CP}^1,$$

imply the condition

$$\deg_z A \mid \text{LCM}\left(\nu'_1(A(z)), \nu'_2(A(z)), \dots, \nu'_s(A(z))\right), \quad z \in \mathbb{CP}^1. \quad \square$$

Corollary 4.5. *Let be a rational function A of degree at least two. Then there exists at most one orbifold \mathcal{O} such that $A : \mathcal{O} \rightarrow \mathcal{O}$ is a covering map between orbifolds.*

Proof. Assume that there exist two such orbifolds $\mathcal{O}_1, \mathcal{O}_2$, and set $\mathcal{O} = \text{LCM}(\mathcal{O}_1, \mathcal{O}_2)$. By formula (17), both $\mathcal{O}_1, \mathcal{O}_2$ have zero Euler characteristic. Moreover, since $A : \mathcal{O} \rightarrow \mathcal{O}$ is a covering map between orbifolds, $\chi(\mathcal{O}) = 0$. However, it is easy to see that whenever $\nu(\mathcal{O}_1)$ and $\nu(\mathcal{O}_2)$ belong to list (29) the equality $\chi(\mathcal{O}) = 0$ is possible only $\mathcal{O}_1 = \mathcal{O}_2$. \square

Theorem 4.6. *Let A be a rational function of degree $d \geq 2$ not conjugated to $z^{\pm d}$ or $\pm T_d$. Then there exists an orbifold \mathcal{O}_0^A such that $A : \mathcal{O}_0^A \rightarrow \mathcal{O}_0^A$ is a minimal holomorphic map and for any orbifold \mathcal{O} such that $A : \mathcal{O} \rightarrow \mathcal{O}$ is a minimal holomorphic map the relation $\mathcal{O} \preceq \mathcal{O}_0^A$ holds.*

Proof. It is enough to show that there exist at most a finite number of orbifolds \mathcal{O} such that $A : \mathcal{O} \rightarrow \mathcal{O}$ is a minimal holomorphic map. Indeed, in this case the orbifold

$$\mathcal{O}_0^A = \text{LCM}(\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_l),$$

where $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_l$ is the complete list of such orbifolds, satisfies the conditions of the theorem by Proposition 2.2.

Let $\mathcal{O}_1, \mathcal{O}_2, \dots$ be a sequence of pairwise distinct orbifolds such that each $A : \mathcal{O}_i \rightarrow \mathcal{O}_i$ is a minimal holomorphic map. Set

$$\mathcal{U}_s = \text{LCM}(\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_s), \quad s \geq 1.$$

Then the maps $A : \mathcal{U}_s \rightarrow \mathcal{U}_s, s \geq 1$, are minimal holomorphic maps between orbifolds by Proposition 4.4. Since $\chi(\mathcal{U}_s) \geq 0$ by Proposition 2.2, it follows from $\mathcal{U}_s \preceq \mathcal{U}_{s+1}$ that if the above sequence is infinite, then for s big enough either $\nu(\mathcal{U}_s) = \{n, n\}$, or $\nu(\mathcal{U}_s) = \{2, 2, n\}$, and $n \rightarrow \infty$ as $s \rightarrow \infty$. However, in this case Theorem 4.3 implies that A is conjugated either to $z^{\pm d}$ or to $\pm T_d$. \square

Since orbifolds \mathcal{O} satisfying $\chi(\mathcal{O}) \geq 0$ are listed in (29), (30), Theorem 4.6 implies in particular that if for a given rational function A there exist two orbifolds \mathcal{O}_1 and \mathcal{O}_2 distinct from the non-ramified sphere such that $A : \mathcal{O}_1 \rightarrow \mathcal{O}_1$ and $A : \mathcal{O}_2 \rightarrow \mathcal{O}_2$ are minimal holomorphic maps between orbifolds, then, up to a small number of exceptions, $c(\mathcal{O}_1) = c(\mathcal{O}_2)$, and either $\nu(\mathcal{O}_1) = \{2, 2, k\}$, $\nu(\mathcal{O}_2) = \{2, 2, l\}$, or $\nu(\mathcal{O}_1) = \{k, k\}, \nu(\mathcal{O}_2) = \{l, l\}$ for some $k \geq 2$ and $l \geq 2$.

Notice that the functions $z^{\pm n}$ and $\pm T_n$ become true covering self-maps between orbifolds if to allow the base Riemann surface to be non-compact ([4]). Namely, it is easy to see that the map $z^{\pm n} : \mathcal{O} \rightarrow \mathcal{O}$ is a covering map for the non-ramified orbifold with the base surface $\mathcal{R} = \mathbb{C} \setminus \{0, \infty\}$, while $\pm T_n : \mathcal{O} \rightarrow \mathcal{O}$ is a covering map for the orbifold defined on $\mathcal{R} = \mathbb{C} \setminus \{\infty\}$ by the condition $\nu(1) = 2, \nu(-1) = 2$. The corresponding functions $\theta_{\mathcal{O}}$ are e^z and $\cos z$. Notice also that $z^{\pm n}$ and $\pm T_n$ along with Lattès maps play a key role in the description of commuting rational functions obtained by Ritt (see [19], [4], [18]).

5 Explicit formulas

5.1 Ramification collections $\{n, n\}$ and $\{2, 2, n\}$

Let \mathcal{O} be an orbifold with $\chi(\mathcal{O}) > 0$. Recall that by Proposition 2.1 and Theorem 2.8 for any minimal holomorphic map $A : \mathcal{O} \rightarrow \mathcal{O}$ there exists a rational function $F : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ and an automorphism $\varphi : \Gamma_{\mathcal{O}} \rightarrow \Gamma_{\mathcal{O}}$ such that the diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{F} & \mathbb{CP}^1 \\ \downarrow \theta_{\mathcal{O}} & & \downarrow \theta_{\mathcal{O}} \\ \mathcal{O} & \xrightarrow{A} & \mathcal{O} \end{array} \quad (65)$$

is commutative, and

$$F \circ \sigma = \varphi(\sigma) \circ F, \quad \sigma \in \Gamma_{\mathcal{O}}. \quad (66)$$

In this subsection, using the link between A and F , we describe rational functions A which are minimal holomorphic maps $A : \mathcal{O} \rightarrow \mathcal{O}$ for orbifolds \mathcal{O} with $\nu(\mathcal{O}) = \{n, n\}$, $n \geq 2$, or $\nu(\mathcal{O}) = \{2, 2, n\}$, $n > 2$. We also describe all *polynomial* minimal holomorphic maps between orbifolds $A : \mathcal{O} \rightarrow \mathcal{O}$ with $\chi(\mathcal{O}) > 0$.

We start from orbifolds \mathcal{O} with $\nu(\mathcal{O}) = \{n, n\}$, $n \geq 2$. To be definite, we normalize \mathcal{O} by the condition

$$\nu(0) = n, \quad \nu(\infty) = n, \quad n \geq 2. \quad (67)$$

Theorem 5.1. *Let \mathcal{O} be an orbifold defined by (67), and A a rational function. Then $A : \mathcal{O} \rightarrow \mathcal{O}$ is a minimal holomorphic map between orbifolds if and only if $A = z^r R^n(z)$, where $R \in \mathbb{C}(z)$, $1 \leq r \leq n-1$, and $\text{GCD}(r, n) = 1$.*

Proof. For \mathcal{O} defined by (67) the corresponding group $\Gamma_{\mathcal{O}}$ is a cyclic group C_n generated by the transformation

$$\alpha : z \rightarrow e^{2\pi i/n} z,$$

and the function $\theta_{\mathcal{O}}$ has the form $\theta_{\mathcal{O}} = z^n$. Since a homomorphism $\varphi : \Gamma_{\mathcal{O}} \rightarrow \Gamma_{\mathcal{O}}$ is an automorphism if and only if (54) holds, we conclude that F satisfies (66) if and only if F/z^r is $\Gamma_{\mathcal{O}}$ -invariant. Thus, $F = z^r R(z^n)$, where $R \in \mathbb{C}(z)$, $1 \leq r \leq n-1$, and $\text{GCD}(r, n) = 1$. Since the function $A = z^r R^n(z)$ makes corresponding diagram (65) commutative, this proves the statement. \square

Denote by \mathfrak{T} the set of rational functions commuting with the transformation

$$\beta : z \rightarrow \frac{1}{z}.$$

Since the equality $G(z)G(1/z) = 1$ implies that $a \in \mathbb{CP}^1$ is a zero of G of order k if and only if $1/a$ is a pole of G of order k , it is easy to see that a rational function G belongs to \mathfrak{T} if and only if

$$G = \pm z^{\pm l_0} \frac{(z - a_1)^{l_1} (z - a_2)^{l_2} \dots (z - a_s)^{l_s}}{(a_1 z - 1)^{l_1} (a_2 z - 1)^{l_2} \dots (a_s z - 1)^{l_s}},$$

where $a_1, a_2, \dots, a_s \in \mathbb{C} \setminus \{0\}$ and $l_1, l_2, \dots, l_s \in \mathbb{N}$. The next statement describes minimal holomorphic maps $A : \mathcal{O} \rightarrow \mathcal{O}$ between orbifolds for \mathcal{O} with $\nu(\mathcal{O}) = \{2, 2, n\}$, normalized by the condition

$$\nu(-1) = 2, \quad \nu(1) = 2, \quad \nu(\infty) = n, \quad n > 2. \quad (68)$$

The case $n = 2$ can be analyzed by the method of the next section.

Theorem 5.2. *Let \mathcal{O} be an orbifold defined by (68) and A a rational function. Then $A : \mathcal{O} \rightarrow \mathcal{O}$ is a minimal holomorphic map between orbifolds if and only if*

$$A = \pm \frac{1}{2} \left(z^r S^n(z) \circ (z + \sqrt{z^2 - 1}) + z^r S^n(z) \circ (z - \sqrt{z^2 - 1}) \right) \quad (69)$$

where $S \in \mathfrak{T}$, $1 \leq r \leq n - 1$, and $\text{GCD}(r, n) = 1$.

Proof. For \mathcal{O} defined by (68) the corresponding group $\Gamma_{\mathcal{O}}$ is a dihedral group D_{2n} generated by the transformations α and β , and the function $\theta_{\mathcal{O}}$ has the form (58). Let F be a function satisfying (65) and (66). Since $n > 2$, equalities (54) and (60) hold.

As in the proof of Theorem 5.1, equality (54) implies that $F = z^r R(z^n)$, where $R \in \mathbb{C}(z)$, $1 \leq r \leq n - 1$, and $\text{GCD}(r, n) = 1$. On the other hand, equality (60) implies that

$$F(1/z) = e^{\frac{2\pi i}{n}k} \frac{1}{F(z)}, \quad 0 \leq k \leq n - 1. \quad (70)$$

Therefore,

$$R(z^n)R(1/z^n) = e^{\frac{2\pi i}{n}k}, \quad 0 \leq k \leq n - 1,$$

implying that $e^{-\frac{\pi i}{n}k} R \in \mathfrak{T}$. Thus, F has the form

$$F = \varepsilon z^r S(z^n), \quad (71)$$

where $S \in \mathfrak{T}$, $\varepsilon^{2n} = 1$, $1 \leq r \leq n - 1$, and $\text{GCD}(r, n) = 1$. Moreover, any such F satisfies (16) for some automorphism $\varphi : \Gamma_{\mathcal{O}} \rightarrow \Gamma_{\mathcal{O}}$.

Finally, since

$$A \circ \frac{1}{2} \left(z^n + \frac{1}{z^n} \right) = \frac{1}{2} \left(z^n + \frac{1}{z^n} \right) \circ \varepsilon z^r S(z^n),$$

it follows from

$$A \circ \frac{1}{2} \left(z^n + \frac{1}{z^n} \right) = A \circ \frac{1}{2} \left(z + \frac{1}{z} \right) \circ z^n$$

and

$$\frac{1}{2} \left(z^n + \frac{1}{z^n} \right) \circ \varepsilon z^r S(z^n) = \frac{1}{2} \left(z + \frac{1}{z} \right) \circ \pm z^r S^n(z) \circ z^n,$$

that

$$A \circ \frac{1}{2} \left(z + \frac{1}{z} \right) = \frac{1}{2} \left(z + \frac{1}{z} \right) \circ \pm z^r S^n(z) = \pm \frac{1}{2} \left(z^r S^n(z) + z^r S^n(z) \circ \frac{1}{z} \right).$$

Substituting now in the both parts of the last equality $z = z + \sqrt{z^2 - 1}$ we obtain (69). \square

The next proposition describes all polynomial minimal holomorphic maps between orbifolds $A : \mathcal{O} \rightarrow \mathcal{O}$ with $\chi(\mathcal{O}) > 0$.

Theorem 5.3. *Let \mathcal{O} be an orbifold with $\chi(\mathcal{O}) > 0$ distinct from the non-ramified sphere, and A a polynomial of degree at least two such that $A : \mathcal{O} \rightarrow \mathcal{O}$ is a minimal holomorphic map between orbifolds. Then either $\nu(\mathcal{O}) = \{n, n\}$, $n \geq 2$, and A is conjugated to $z^r R^n(z)$, where $R \in \mathbb{C}[z]$, $1 \leq r \leq n - 1$, and $\text{GCD}(r, n) = 1$, or $\nu(\mathcal{O}) = (2, 2, n)$, $n \geq 2$, and A is conjugated to $\pm T_m$, where $\text{GCD}(m, n) = 1$.*

Proof. Since A is a polynomial, it follows from (65) that the set $\theta_{\mathcal{O}}^{-1}\{\infty\}$ is completely invariant with respect to F , implying by the Riemann-Hurwitz formula that $\theta_{\mathcal{O}}^{-1}\{\infty\}$ contains at most two points. Furthermore, if $\theta_{\mathcal{O}}^{-1}\{\infty\}$ contains one point, then F is conjugate to a polynomial, while if $\theta_{\mathcal{O}}^{-1}\{\infty\}$ contains two points, then F is conjugate to $z^{\pm n}$ (see e. g. [20], Theorem 1.6).

Since the cardinality of $\theta_{\mathcal{O}}^{-1}\{\infty\}$ equals the length of an orbit of $\Gamma_{\mathcal{O}}$, and $\Gamma_{\mathcal{O}}$ is one of five finite rotation groups of the sphere, the inequality $|\theta_{\mathcal{O}}^{-1}\{\infty\}| \leq 2$ implies that $\Gamma_{\mathcal{O}}$ is either C_n or D_{2n} (see e. g. [7]). Furthermore, if $\Gamma_{\mathcal{O}} = C_n$, then $\nu(\mathcal{O}) = \{n, n\}$, $n \geq 2$, and Theorem 5.1 implies easily that A is conjugated to $z^r R^n(z)$, where $R \in \mathbb{C}[z]$, $1 \leq r \leq n - 1$, and $\text{GCD}(r, n) = 1$.

Assume now that $\Gamma_{\mathcal{O}} = D_{2n}$. Then $\nu(\mathcal{O}) = \{2, 2, n\}$, $n \geq 2$, and without loss of generality we may assume that equality (58) holds. Since equality $F^{-1}\{0, \infty\} = \{0, \infty\}$ implies equality (56), arguing now as in the proof of Theorem 4.3 we conclude that A is conjugated to $\pm T_m$, where $\text{GCD}(m, n) = 1$. \square

5.2 Arbitrary ramification collections

In this subsection we describe an approach to the description of minimal holomorphic maps $A : \mathcal{O} \rightarrow \mathcal{O}$ for arbitrary \mathcal{O} with $\chi(\mathcal{O}) > 0$ basing on the link between such maps and rational functions F commuting with $\Gamma_{\mathcal{O}}$. Clearly, any such F satisfies condition (66) for the *identical* automorphism $\varphi = id$. For example, a rational function $A = z^r R^n(z)$ from Theorem 5.1 commutes with $\Gamma_{\mathcal{O}} = C_n$ if and only if $r = 1$. Denote by $Out(\Gamma_{\mathcal{O}})$ the outer automorphism group of $\Gamma_{\mathcal{O}}$, and by $d_{\mathcal{O}}$ the order of $Out(\Gamma_{\mathcal{O}})$.

Lemma 5.4. *Let $A : \mathcal{O} \rightarrow \mathcal{O}$ be a minimal holomorphic map for some orbifold with $\chi(\mathcal{O}) > 0$. Then there exists a rational function \tilde{F} commuting with $\Gamma_{\mathcal{O}}$ such that the diagram*

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{\tilde{F}} & \mathbb{CP}^1 \\ \downarrow \theta_{\mathcal{O}} & & \downarrow \theta_{\mathcal{O}} \\ \mathcal{O} & \xrightarrow{A \circ d_{\mathcal{O}}} & \mathcal{O} \end{array} \quad (72)$$

is commutative.

Proof. Let F be an arbitrary rational functions satisfying (65). Recall that F is defined up to the composition $g \circ F$, where $g \in \Gamma_{\mathcal{O}}$. Furthermore, it is easy to see that for $g \in \Gamma_{\mathcal{O}}$ the change $F \rightarrow g \circ F$ corresponds to the change $\varphi \rightarrow g \circ \varphi \circ g^{-1}$. In particular, if φ is inner, then for an appropriate g the automorphism $g \circ \varphi \circ g^{-1}$ is identical, or equivalently the function $g \circ F$ commutes with $\Gamma_{\mathcal{O}}$. Since the automorphism $\varphi^{od_{\mathcal{O}}}$ is inner, it follows now from the equalities

$$A^{\circ n} \circ \theta_{\mathcal{O}} = \theta_{\mathcal{O}} \circ F^{\circ n}, \quad n \geq 1,$$

and

$$F^{\circ n} \circ \sigma = \varphi^{\circ n}(\sigma) \circ F^{\circ n}, \quad \sigma \in \Gamma_{\mathcal{O}},$$

that there exists $g \in \Gamma_{\mathcal{O}}$ such that the function $\tilde{F} = g \circ F^{od_{\mathcal{O}}}$ satisfies (72) and commutes with $\Gamma_{\mathcal{O}}$. \square

Since $Out(S_4)$ is trivial, Lemma 5.4 reduces the description of minimal holomorphic maps $A : \mathcal{O} \rightarrow \mathcal{O}$ for orbifolds \mathcal{O} with $\nu(\mathcal{O}) = \{2, 3, 4\}$ to the description of rational functions commuting with S_4 . On the other hand, since

$$Out(A_5) = Out(A_4) = \mathbb{Z}/2\mathbb{Z},$$

it follows from Lemma 5.4 that in order to describe all minimal holomorphic maps $A : \mathcal{O} \rightarrow \mathcal{O}$ with $\nu(\mathcal{O}) = \{2, 3, 3\}$ or $\nu(\mathcal{O}) = \{2, 3, 5\}$ it is enough to describe the maps corresponding to functions commuting with $\Gamma_{\mathcal{O}}$ and “compositional square roots” of such maps. The method for describing rational functions commuting with finite automorphism groups of \mathbb{CP}^1 was given in [3]. We overview it below.

Identify a rational function f of degree d with its *dual 1-form* as follows. Take a representation $f = f_1/f_2$, where f_1 and f_2 are polynomials without multiple roots, denote by F_i the homogenization of f_i to the degree d , and set

$$\omega = -F_2 dx + F_1 dy.$$

It is clear that the form ω is defined up to a multiplication by $\lambda \in \mathbb{C} \setminus \{0\}$, and forms ω_1 and ω_2 represent the same function if and only if $\omega_2 = \lambda \omega_1$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. Under this identification the function $\mu^{-1} \circ f \circ \mu$, where

$$\mu = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \alpha, \beta, \gamma, \delta \in \mathbb{C},$$

is identified with the pullback $\mu'^* \omega$, where

$$\mu' : (x, y) \longrightarrow (\alpha x + \beta y, \gamma x + \delta y).$$

Thus, the problem of describing of rational functions commuting with a group Γ reduces to the problem of describing of forms ω such that for any $\mu \in \Gamma$ the equality

$$\mu'^* \omega = \chi(\mu) \omega,$$

holds for some $\chi(\mu) \in \mathbb{C}$. On the other hand, it was shown in [3], that a 1-form of degree d satisfies this condition if and only if

$$\omega = U(x, y)\lambda + dV(x, y), \quad (73)$$

where U and V are invariant homogeneous polynomials with the same character, $\deg U = \deg V + 2$, and λ stands for the form

$$\lambda = -ydx + xdy.$$

It is easy to see that the function f corresponding to form (73) is obtained by setting $z = x/y$ in

$$\frac{xU(x, y) + \frac{\partial V}{\partial y}(x, y)}{yU(x, y) - \frac{\partial V}{\partial x}(x, y)}. \quad (74)$$

Notice that since 0 is a form of every degree, U and V can be equal zero. In particular, for any homogeneous polynomial V we obtain a function commuting with Γ setting $z = x/y$ in

$$-\frac{\frac{\partial V}{\partial y}(x, y)}{\frac{\partial V}{\partial x}(x, y)}. \quad (75)$$

On the other hand, if $V = 0$, then for any U formula (74) leads to the same function $f = z$.

Let us illustrate the above considerations by finding explicitly all minimal holomorphic maps $A : \mathcal{O} \rightarrow \mathcal{O}$ for orbifolds with $\mathcal{O} = \{2, 3, 3\}$ corresponding to forms (73) of degree ≤ 7 . According to Klein [7], homogenous polynomials for the corresponding group $\Gamma = A_4$ are polynomials in

$$\begin{aligned} \Phi &= x^4 + 2i\sqrt{3}x^2y^2 + y^4, \\ \Psi &= x^4 - 2i\sqrt{3}x^2y^2 + y^4, \\ t &= xy(x^4 - y^4). \end{aligned}$$

Furthermore, t is absolutely invariant, while Φ and Ψ are invariant with characters χ_Φ and χ_Ψ whose product is the trivial character. Thus, all forms (73) of degree ≤ 6 are obtained from (75) for V equal Φ , Ψ , or t . The corresponding rational functions commuting with $\Gamma = A_4$ are

$$\begin{aligned} F_1 &= -\frac{i\sqrt{3}z^2 + 1}{z(i\sqrt{3} + z^2)}, \\ F_2 &= -\frac{i\sqrt{3}z^2 - 1}{z(i\sqrt{3} - z^2)}, \\ F_3 &= -\frac{z(z^4 - 5)}{5z^4 - 1}. \end{aligned}$$

For the degree seven we obtain a one-parameter series setting in (73)

$$U = ct, \quad c \in \mathbb{C}, \quad V = \Phi\Psi.$$

In order to obtain the corresponding generalized Lattès map in a compact form, it is convenient to rescale this parametrization setting $c = 8i\sqrt{3}a$, $a \in \mathbb{C}$, so that

$$F_4 = \frac{1}{z} \left(\frac{3az^6 - 7iz^4\sqrt{3} - 3az^2 - i\sqrt{3}}{i\sqrt{3}z^6 + 3az^4 + 7i\sqrt{3}z^2 - 3a} \right).$$

The generalized Lattès maps corresponding to F_i , $1 \leq i \leq 4$, are

$$L_1 = \frac{27x}{(4x-1)^3},$$

$$L_2 = -\frac{(x-4)^3}{27x^2},$$

$$L_3 = -\frac{(5x-4)^3}{x^2(4x-5)^3},$$

and

$$L_4 = z \left(\frac{(a-1)^4 z^2 - 2(a-1)(a^3 - 3a^2 - 9a - 21)z + (a-7)(a+1)^3}{(a+7)(a-1)^3 z^2 - 2(a+1)(a^3 + 3a^2 - 9a + 21)z + (a+1)^4} \right)^3.$$

The functions L_i , $1 \leq i \leq 4$, and F_i , $1 \leq i \leq 4$, are related by the commutative diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{F_i} & \mathbb{CP}^1 \\ \downarrow \theta_{\mathcal{O}} & & \downarrow \theta_{\mathcal{O}} \\ \mathbb{CP}^1 & \xrightarrow{L_i} & \mathbb{CP}^1, \end{array} \quad (76)$$

where \mathcal{O} is normalized by the condition

$$\nu(0) = 3, \quad \nu(1) = 2, \quad \nu(\infty) = 3,$$

and the function

$$\theta_{\mathcal{O}} = \frac{(z^4 + 2i\sqrt{3}z^2 + 1)^3}{(z^4 - 2i\sqrt{3}z^2 + 1)^3}$$

is obtained from Ψ^3/Φ^3 by setting $z = x/y$.

Of coarse, the fact that the functions $L_i : \mathcal{O} \rightarrow \mathcal{O}$, $1 \leq i \leq 4$, are indeed minimal holomorphic maps between orbifolds can be checked directly. Say, it follows from the definition of L_4 that (9) holds for any point z such that $L_4(z) = \infty$, since all points of $L_4^{-1}\{\infty\}$ distinct from ∞ have the multiplicity divisible by 3, while the multiplicity of ∞ is one. The same is true for points z with $L_4(z) = 0$. For points z with $L_4(z) = 1$ formula (9) follows from the formula

$$L_4 - 1 = (z-1) \frac{((a-1)^6 z^3 - (3a^3 + 3a^2 + 45a + 109)(a-1)^3 z^2 + (3a^3 - 3a^2 + 45a - 109)(a+1)^3 z - (a+1)^6)^2}{((a+7)(a-1)^3 z^2 - 2(a+1)(a^3 + 3a^2 - 9a + 21)z + (a+1)^4)^3}.$$

Finally, for points z with $L_4(z) \notin \{0, 1, \infty\}$ condition (9) holds since

$$L_4\{0, 1, \infty\} \subseteq \{0, 1, \infty\}$$

implies that for such z the both parts of (9) equal one.

In conclusion, let us mention two open questions. First, we do not know any practical method for describing “compositional roots” of minimal holomorphic maps corresponding to functions F commuting with $\Gamma_0 = A_4$ or $\Gamma_0 = A_5$ (and whether such roots really exist). Second, we do not know whether the degree of a function commuting with a finite group $\Gamma \subset \text{Aut}(\mathbb{CP}^1)$ is always equal to the degree of the corresponding form reduced by one. In other words, whether the numerator and denominator of (74) for U and V satisfying conditions of the Doyle and McMullen theorem are always coprime. This last question naturally arises while attempting to classify all minimal holomorphic maps up to a fixed degree.

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